Data Structures – Week #5

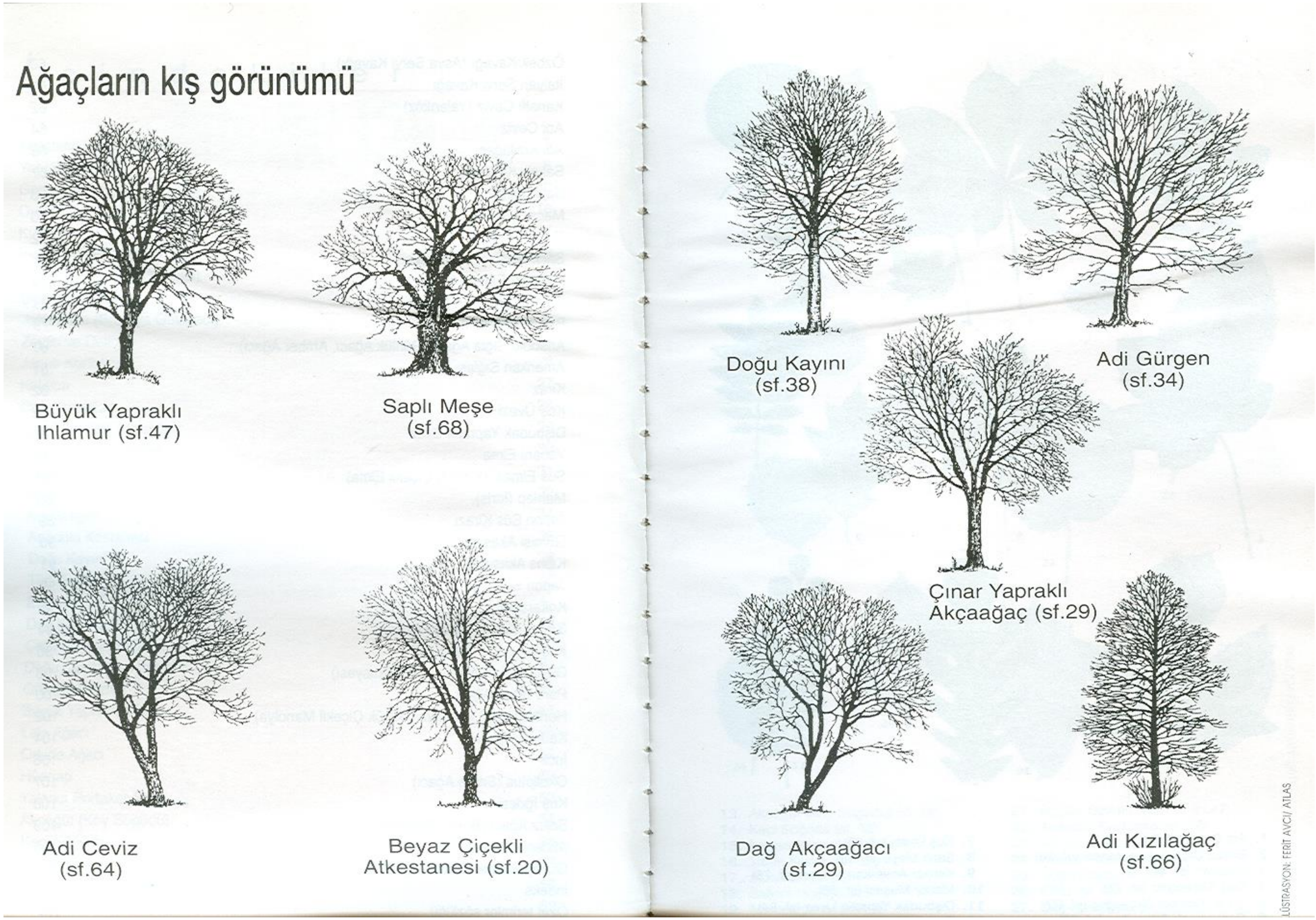
Trees (Ağaçlar)

Trees (Ağaçlar)

Toros Göknarı

Avrupa Göknarı

Trees (Ağaçlar)



Outline

- Trees
- Definitions
- Implementation of Trees
- Binary Trees
- Tree Traversals & Expression Trees
- Binary Search Trees

Definition of a Tree

- Definition:

  A ***tree*** is a collection of nodes (vertices) where the collection may be empty. Otherwise, the collection consists of a distinguished node *r*, called the ***root***, and zero or more non-empty (sub)trees $T_1, \ldots, T_k$, each of whose roots are connected by a *directed edge* to *r*.

More definitions

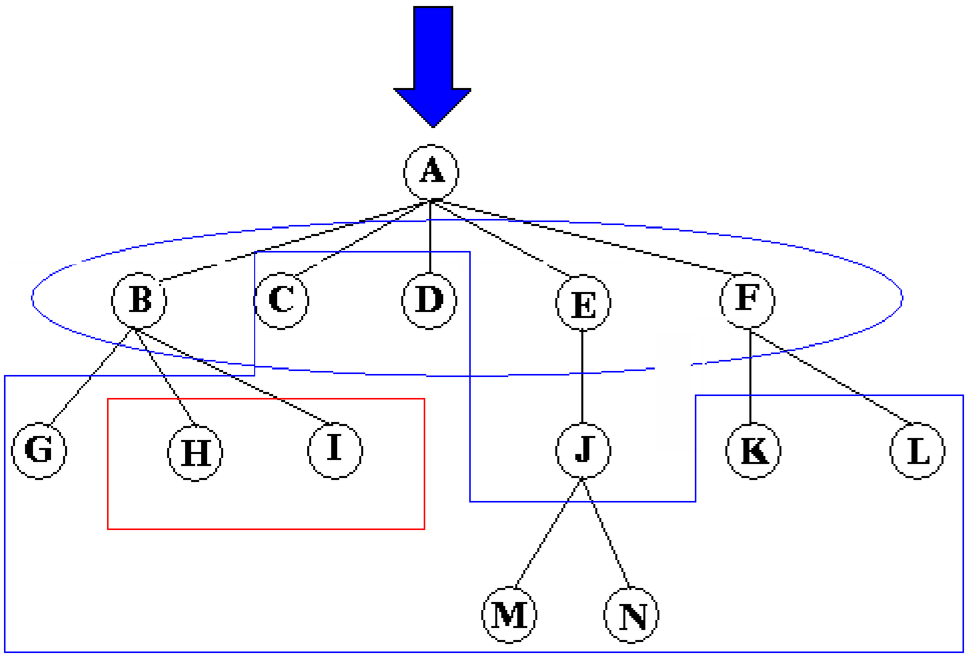
- ***In-degree*** of a vertex is the number of edges arriving at that vertex.
- ***Out-degree*** of a vertex is the number of edges leaving that vertex.
- Nodes with out-degree=0 (no children) are called the ***leaves*** of the tree.
- ***Root*** is the only node in the tree with in-degree=0.
- The ***child*** *C* of a node *A* is the node that is connected to *A* by a single edge. Then, *A* is the ***parent*** of *C*.
- ***Grandparent*** and ***grandchild*** relations can be defined in the same manner.
- Nodes with the same parent are called ***siblings***.

More definitions

- A ***path*** from a node n_1 to n_k is defined as a sequence of nodes $n_1, \ldots, n_k$ such that n_i is the parent of n_{i+1} for $1 \leq i < k$.
  - The ***length*** of this path is the number of edges on the path, i.e., $k-1$.
  - There is **exactly** one path from the root to each node.
- The ***depth*** of any node n_i is the *length of the unique path from the root to* n_i.
  - The depth of root is 0.
- The ***height*** of any node n_i is the *length of the longest path from* n_i *to a leaf.*

A General Tree Example

- *Root*
  - **A**
- *Children* of A
  - **B**, **C**, **D**, **E**, **F**.
- Leaves of the tree
  - **C**, **D**, **G**, **H**, **I**, **K**, **L**, **M**, and **N**.
- Siblings of **G**
  - **H and I**.



Remarks

The ***height*** of all *leaves* are 0.

The ***height*** of a *tree* is the *height of its root*.

The height of the above tree is 3.

The height of C is 0.

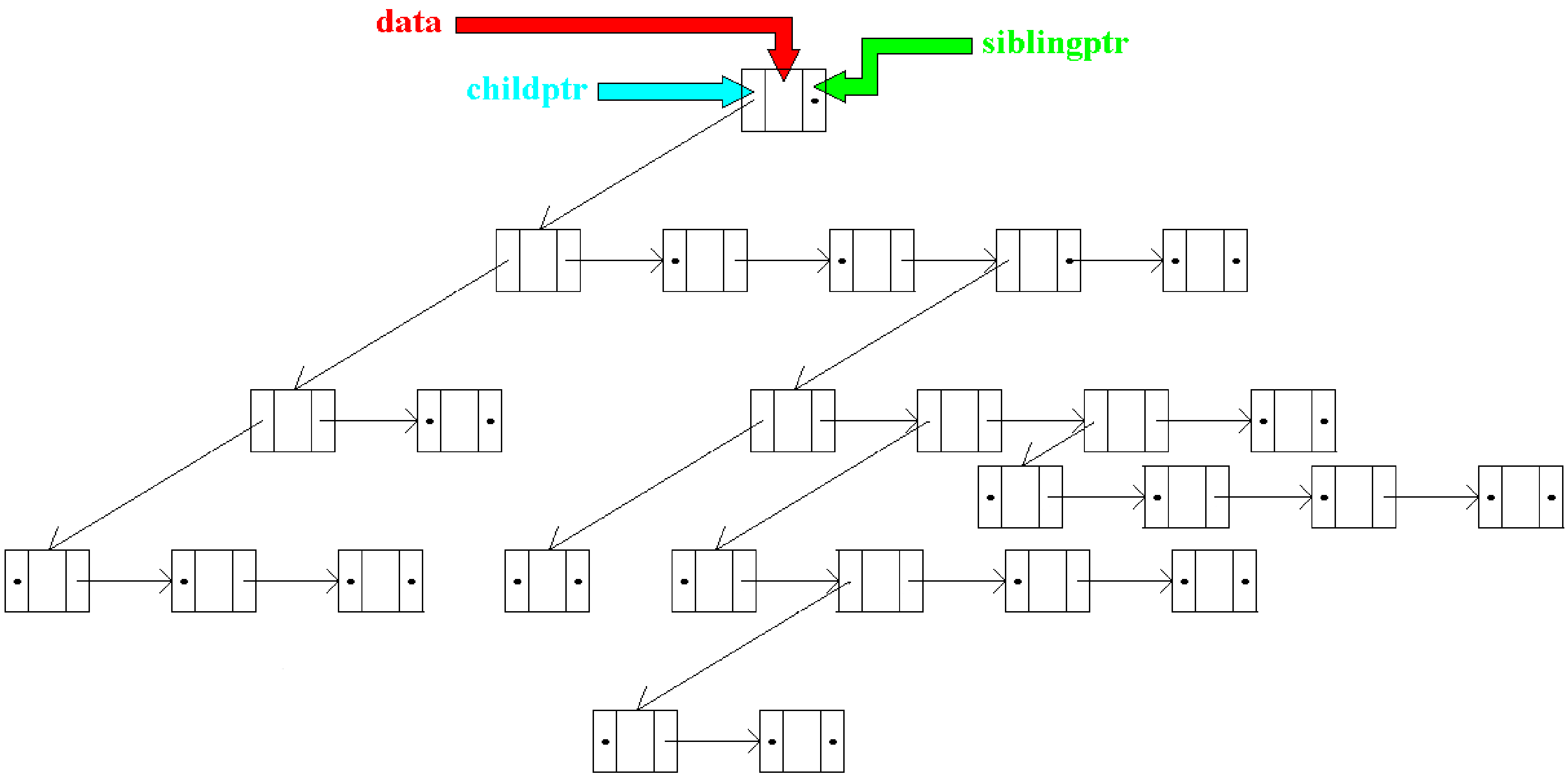
If there is a ***path*** from n_1 to n_2, then n_1 is an *ancestor* of n_2 and n_2 is a *descendant* of n_1 .

Implementation of Trees

- One way to implement a tree would be to have a pointer in each node to each child, besides the data it holds. This is infeasible!
- Q:Why?
- A:The number of children of each node is variable, and hence, unknown.
- Another option would be to keep the children in a linked list where the first node of the list is pointed to by the pointer in the parent node as a header.
- A typical tree node declaration in C would be as follows:

```
struct TNode_type {
      int data;
      struct TNodetype *childptr, *siblingptr;
}
```

A General Tree Implementation

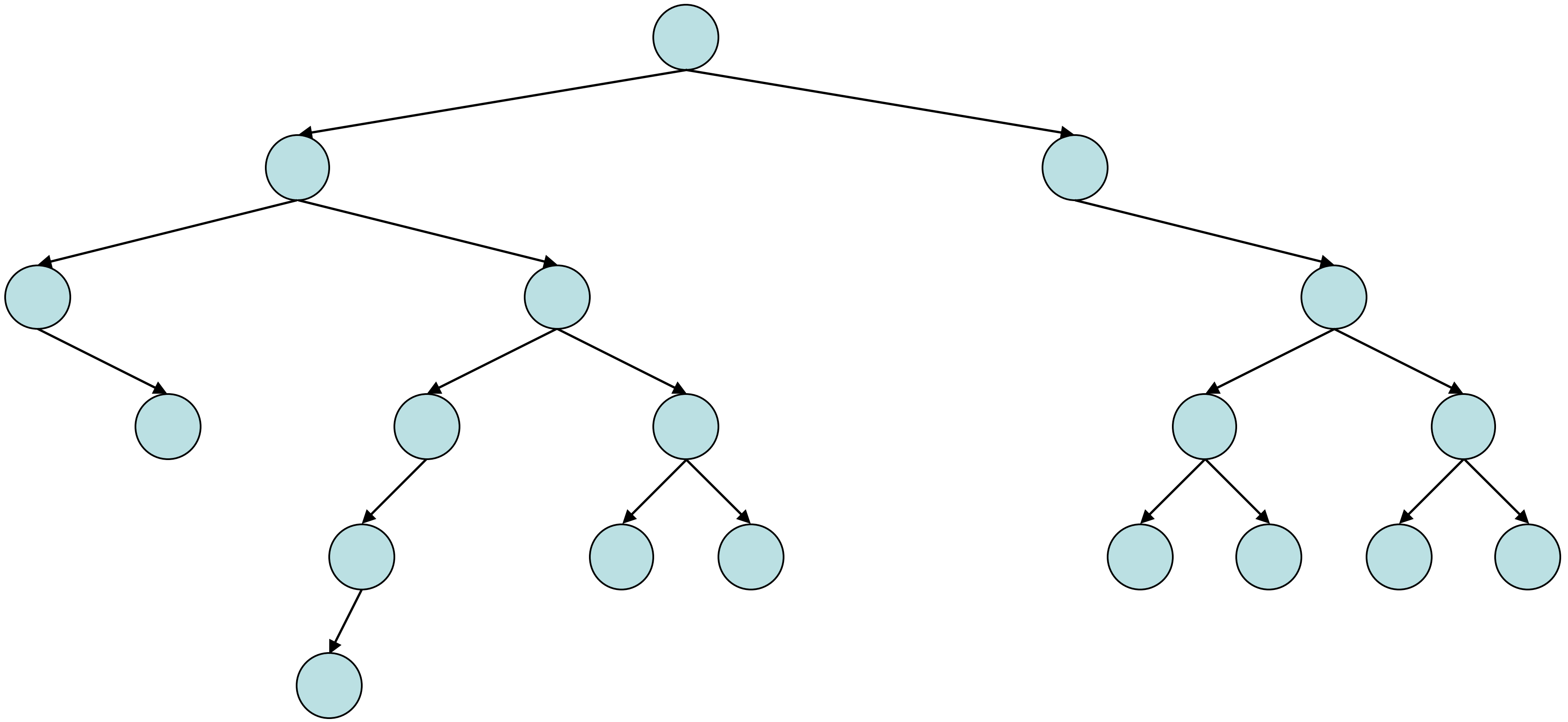


Binary Trees (BTs)

- A *binary tree* is a tree of nodes *with at most two children*.
- Declaration of a typical node in a binary tree
  - struct BTNodeType {

```
    infoType *data;
    struct BTNodeType *left;
    struct BTNodeType *right;
}
```

A Binary Tree Example



Tree Traversals

- A tree can be traversed recursively in three different ways:
  - in-order traversal (left-node-right or LNR)
    - First recursively visit the left subtree.
    - Next process the data in the current node.
    - Finally, recursively visit the right subtree.
  - pre-order traversal (node-left-right or NLR)
  - post-order traversal (left-right-node or LRN)

Recursive In-order (LNR) Traversal

```
void in-order(BTNodeType *p) {
   if (p!=null){
        in-order(p->left);
        operate(p);
        in-order(p->right);
   }
}
```

Recursive Pre-order (NLR) Traversal

```
void pre-order(BTNodeType *p) {
  if (p!=null){
      operate(p);
      pre-order(p->left);
      pre-order(p->right);
  }
}
```

Recursive Post-order (LRN) Traversal

```
void post-order(BTNodeType *p) {
  if (p!=null){
      post-order(p->left);
      post-order(p->right);
      operate(p);
  }
}
```

Expression Trees

- Prefix, postfix and infix formats of arithmetic expressions

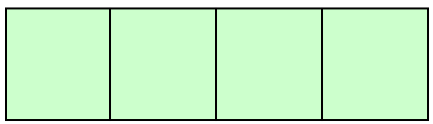
Infix	Postfix	Prefix
A+B	AB+	+AB
A/(B+C)	ABC+/	/A+BC
A/B+C	AB/C+	+/ABC
A-B*C+D/(E+F)	ABC*-DEF+/+	+-A*BC/D+EF
A*((B+C)/(D-E)+F)-G/(H-I)	ABC+DE-/F+*GHI-/-	-*A+/+BC-DEF/G-HI

Construction of an Expression Tree from Postfix Expressions

- Initialize an empty ***stack of subtrees***
- Repeat
  - Get token;
  - if token is an operand
    - Push it as a subtree
  - else
    - pop the last two subtrees and form & push a subtree such that root is the current operator and left and right operands are the former and latter subtrees, respectively
- Until the end of arithmetic expression

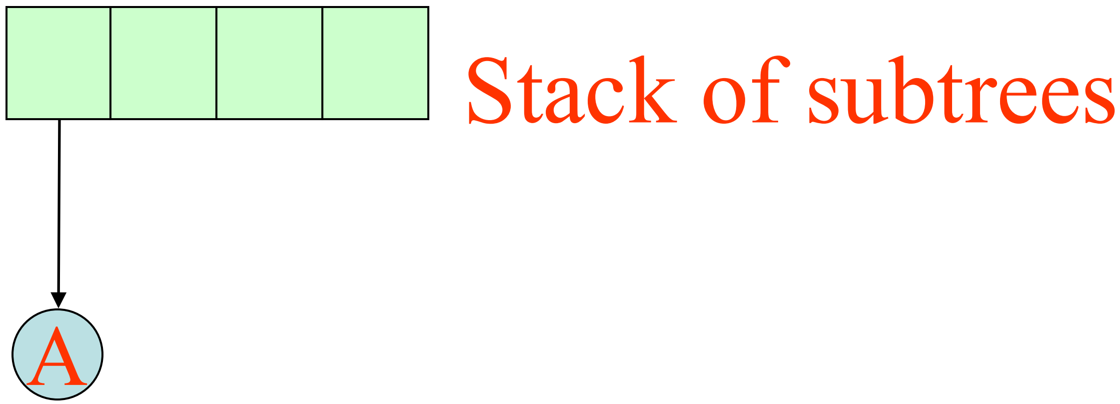
Example to Construction of Expression Trees

Empty Stack of subtrees



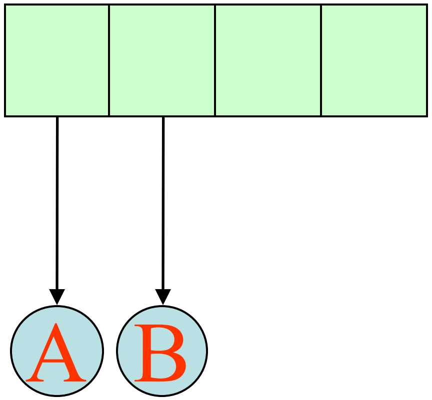
ABC+DE-/F+*GHI-/-

Example to Construction of Expression Trees - 1



ABC+DE-/F+*GHI-/-

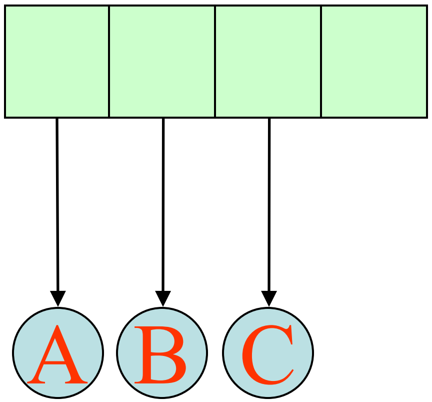
Example to Construction of Expression Trees - 2



Stack of subtrees

ABC+DE-/F+*GHI-/-

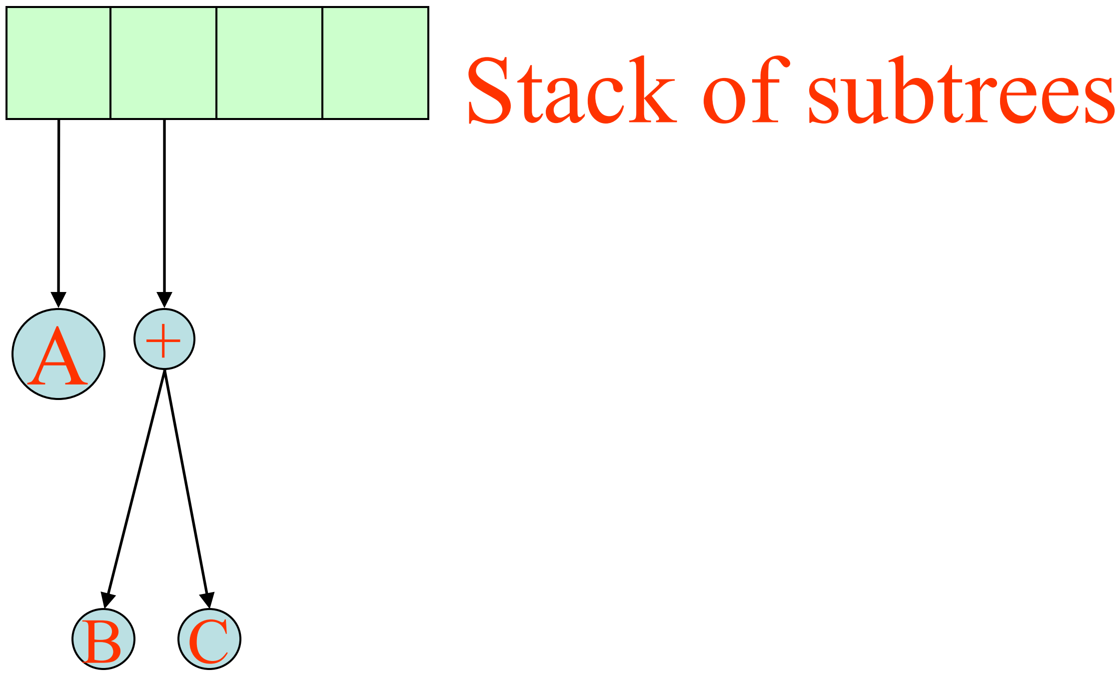
Example to Construction of Expression Trees - 3



Stack of subtrees

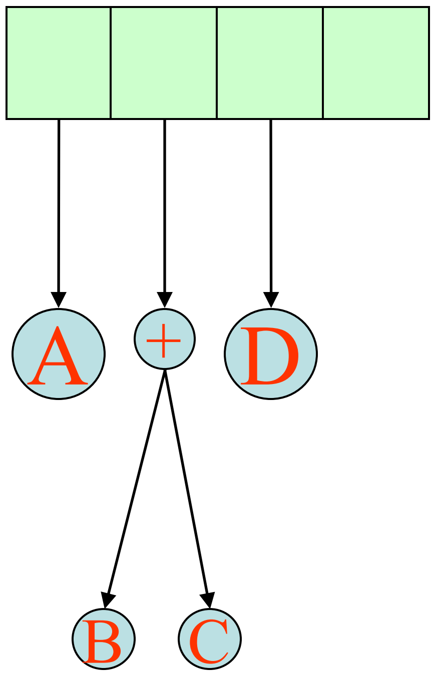
ABC+DE-/F+*GHI-/-

Example to Construction of Expression Trees - 4



ABC+DE-/F+*GHI-/-

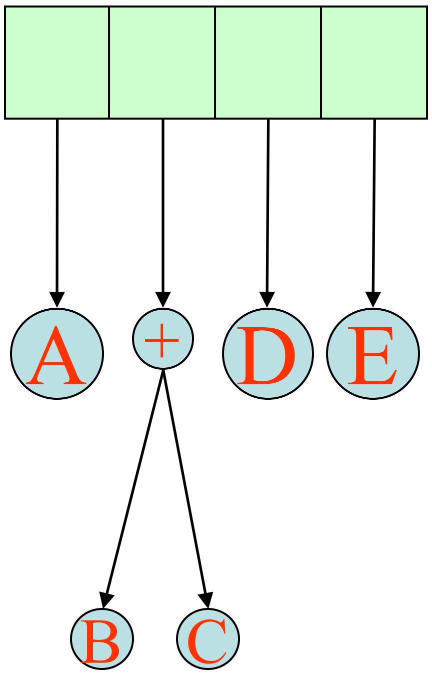
Example to Construction of Expression Trees - 5



Stack of subtrees

ABC+DE-/F+*GHI-/-

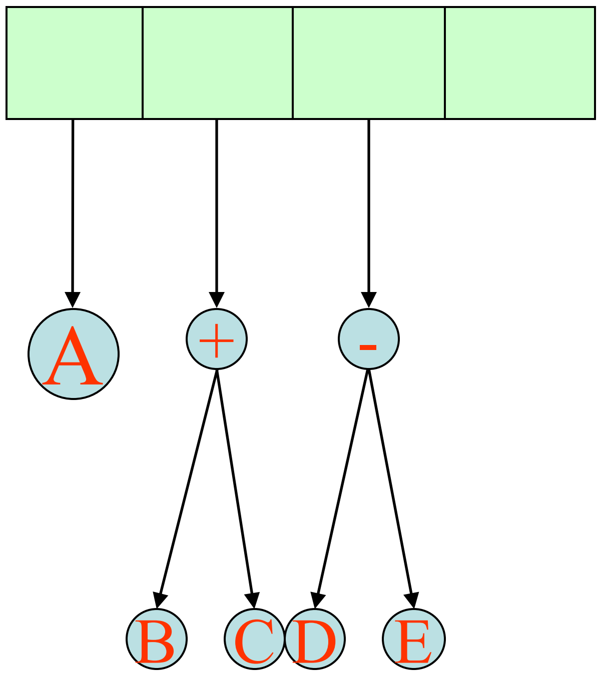
Example to Construction of Expression Trees - 6



Stack of subtrees

ABC+DE-/F+*GHI-/-

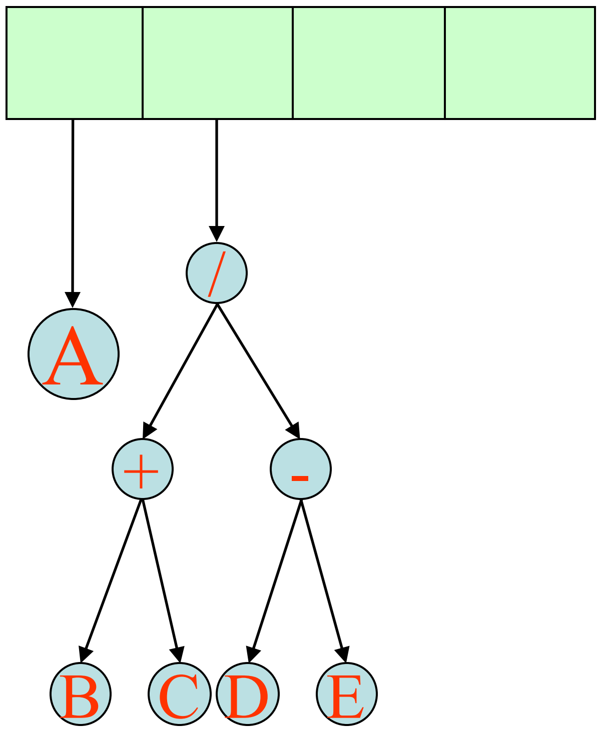
Example to Construction of Expression Trees - 7



Stack of subtrees

ABC+DE-/F+*GHI-/-

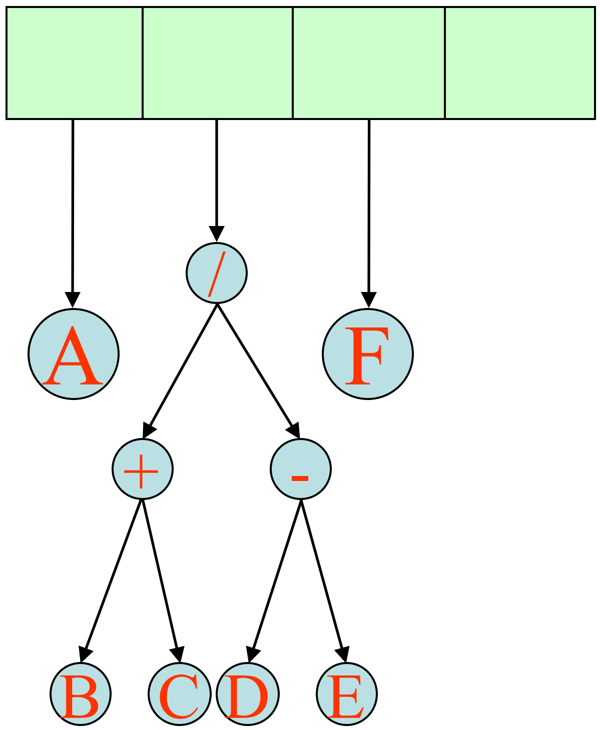
Example to Construction of Expression Trees - 8



Stack of subtrees

ABC+DE-/F+*GHI-/-

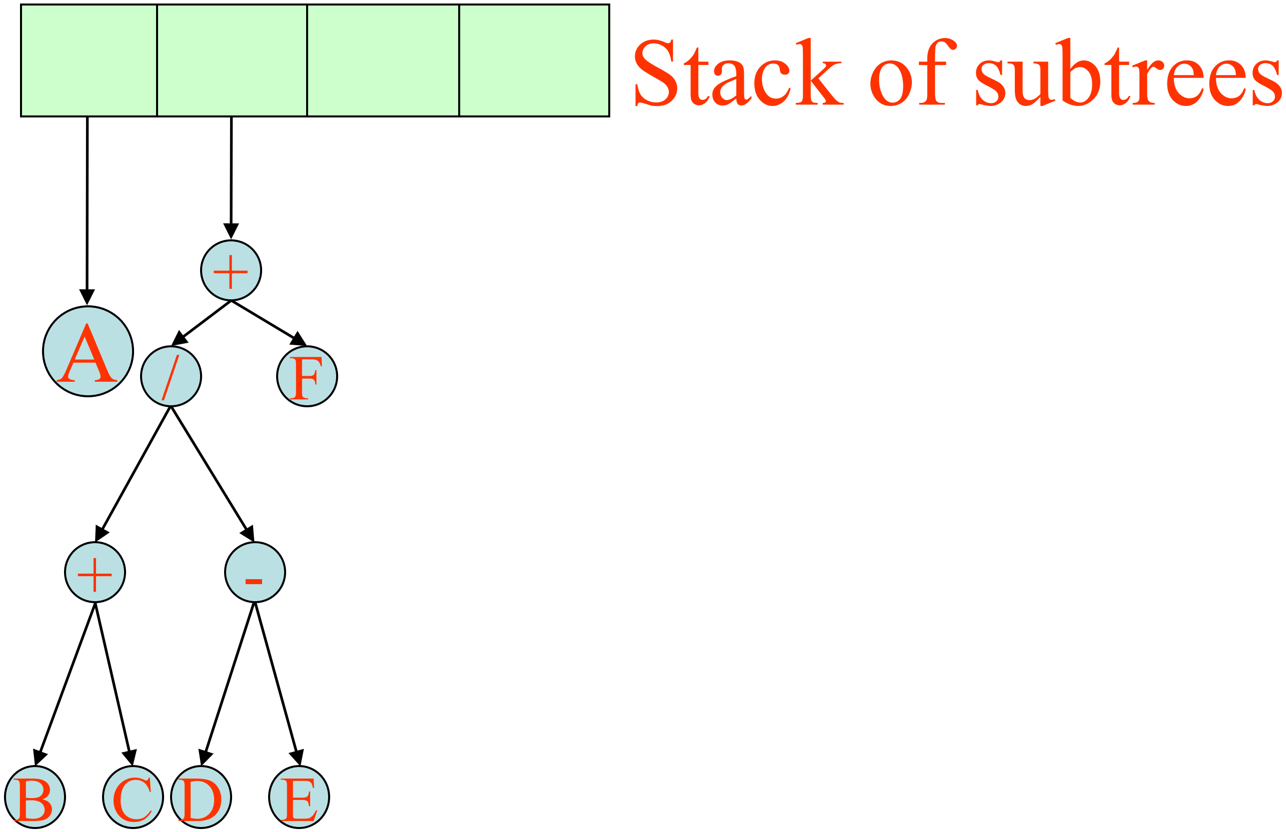
Example to Construction of Expression Trees - 9



Stack of subtrees

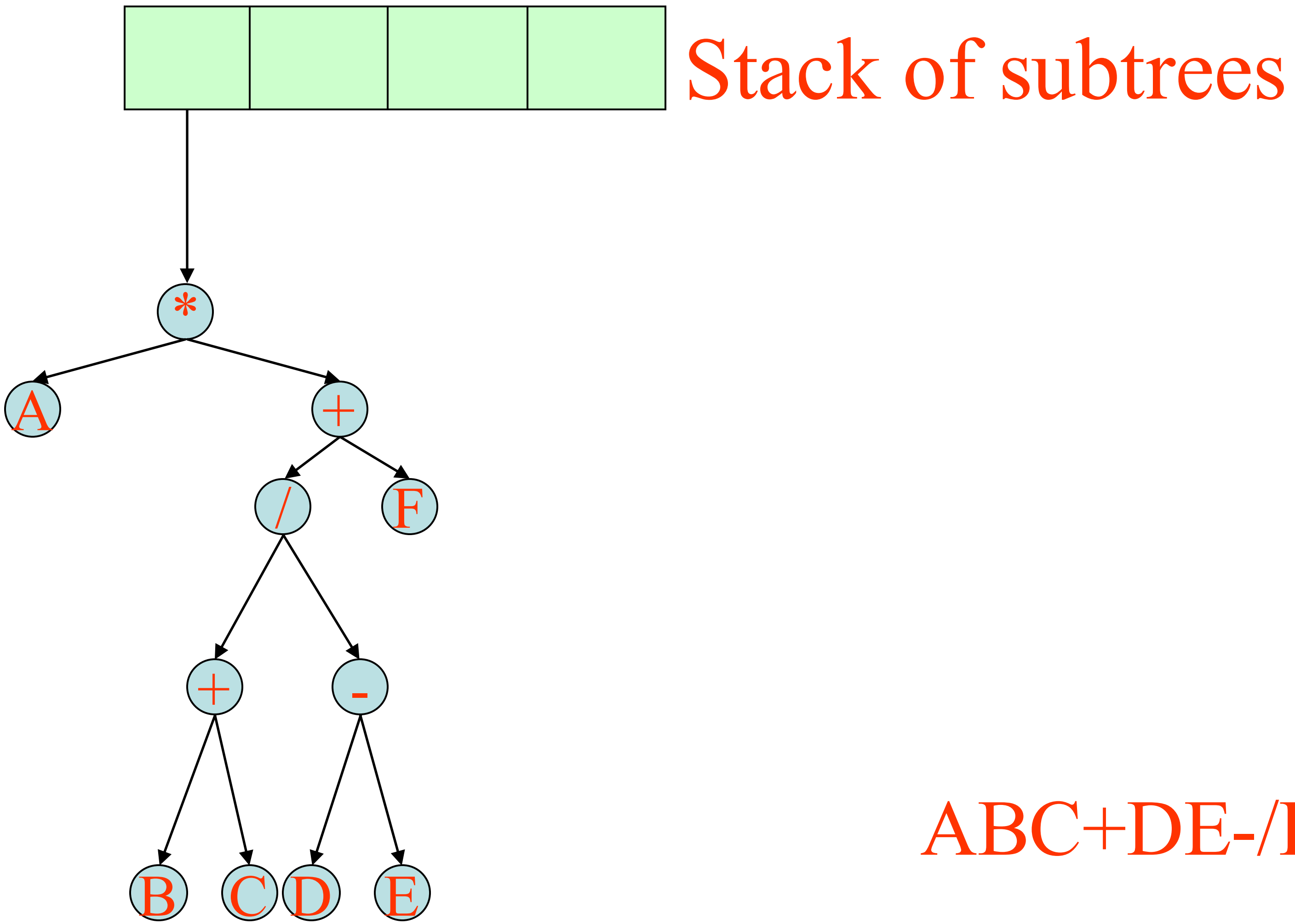
ABC+DE-/F+*GHI-/-

Example to Construction of Expression Trees - 10



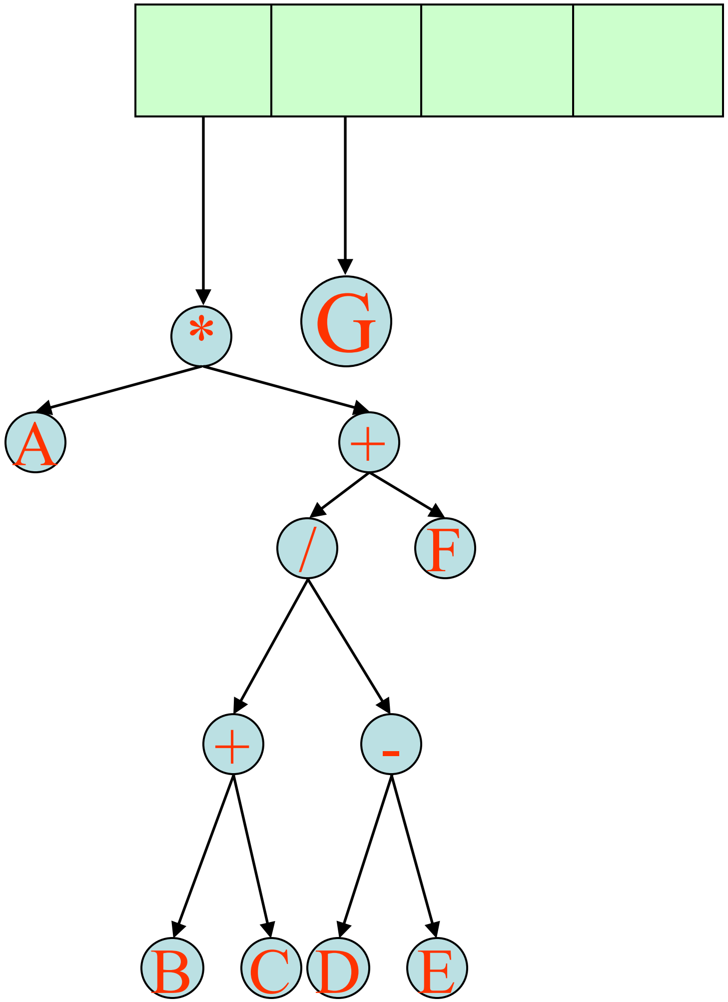
ABC+DE-/F+*GHI-/-

Example to Construction of Expression Trees - 11



ABC+DE-/F+*GHI-/-

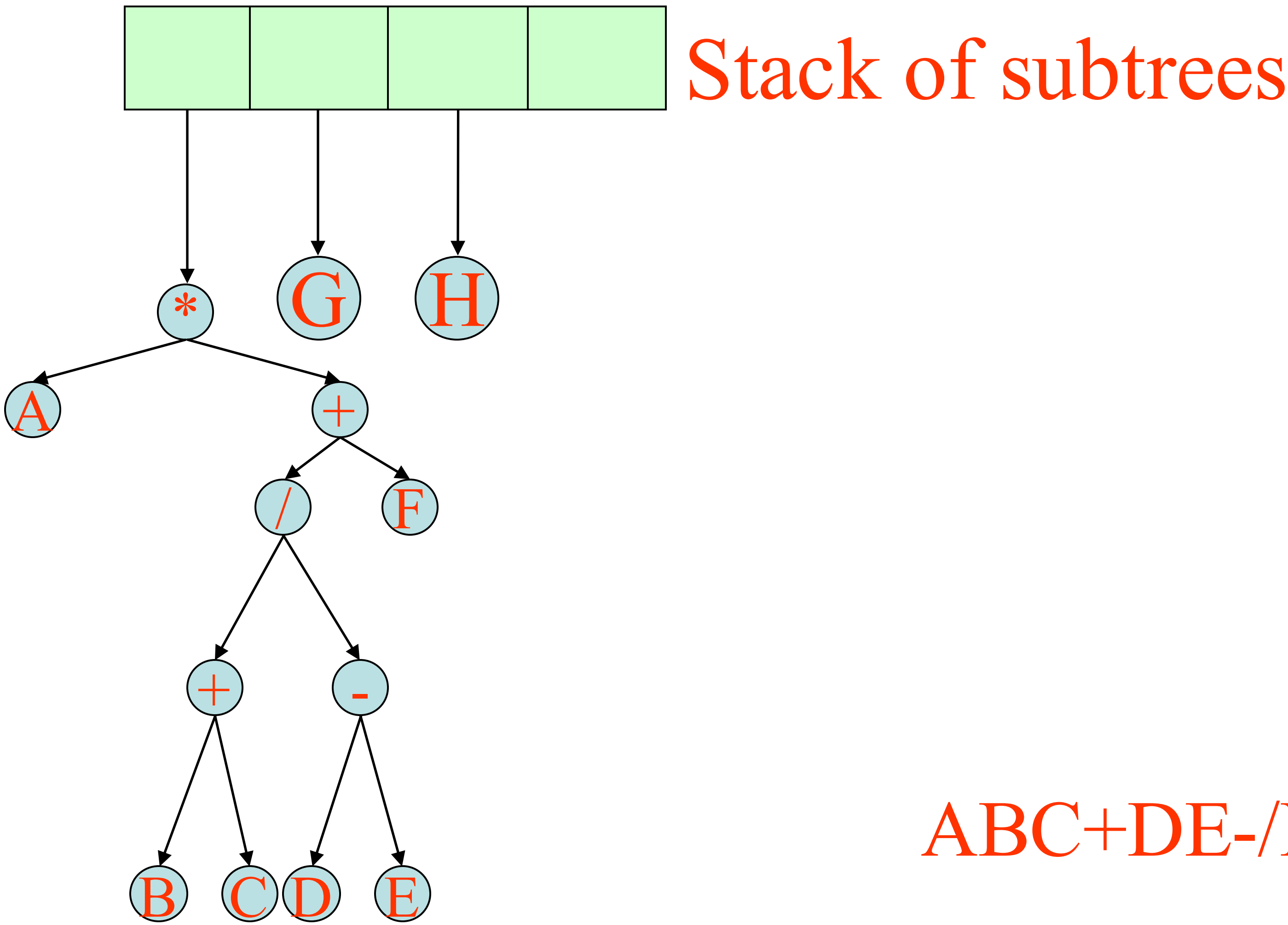
Example to Construction of Expression Trees - 12



Stack of subtrees

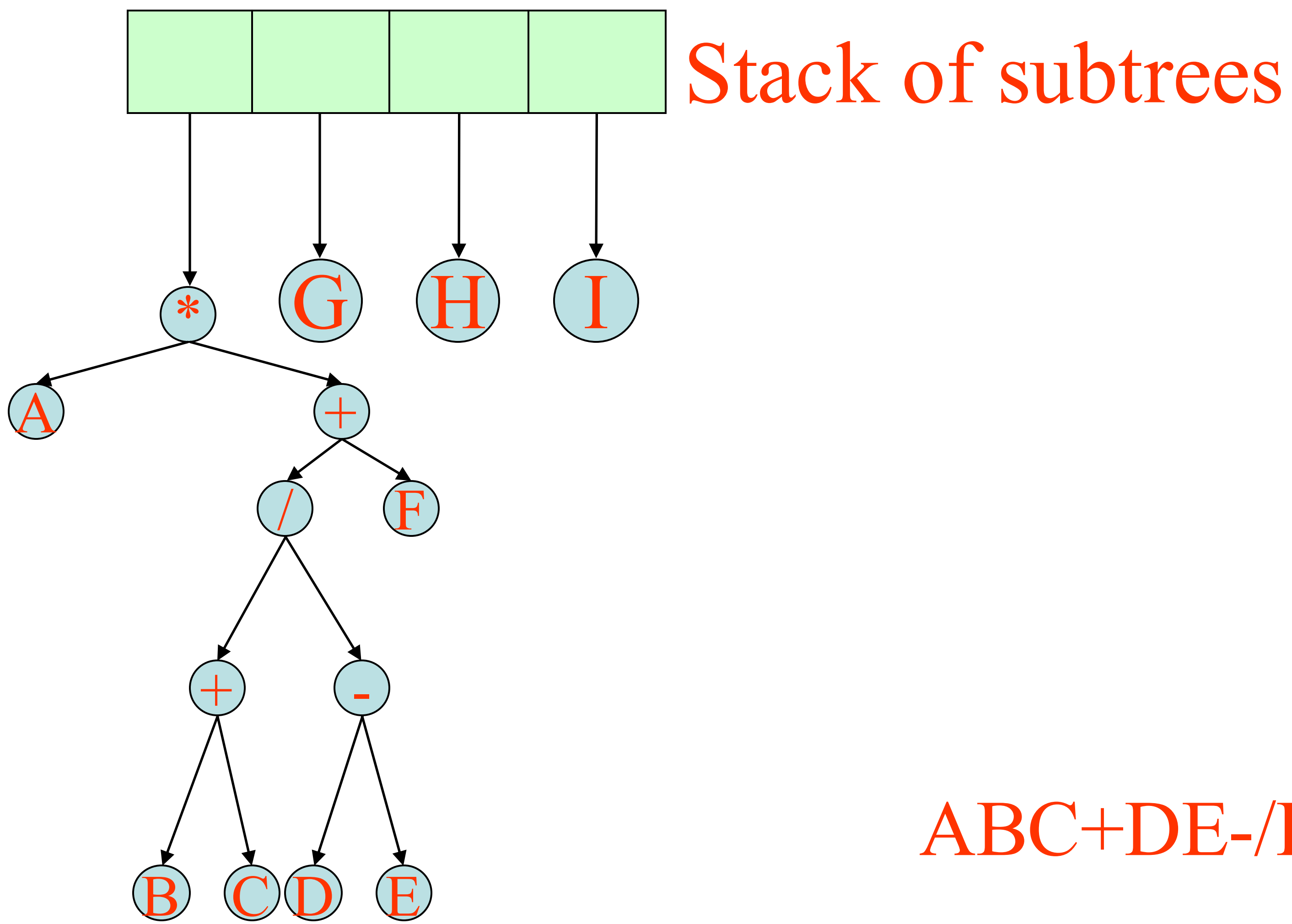
ABC+DE-/F+*GHI-/-

Example to Construction of Expression Trees - 13



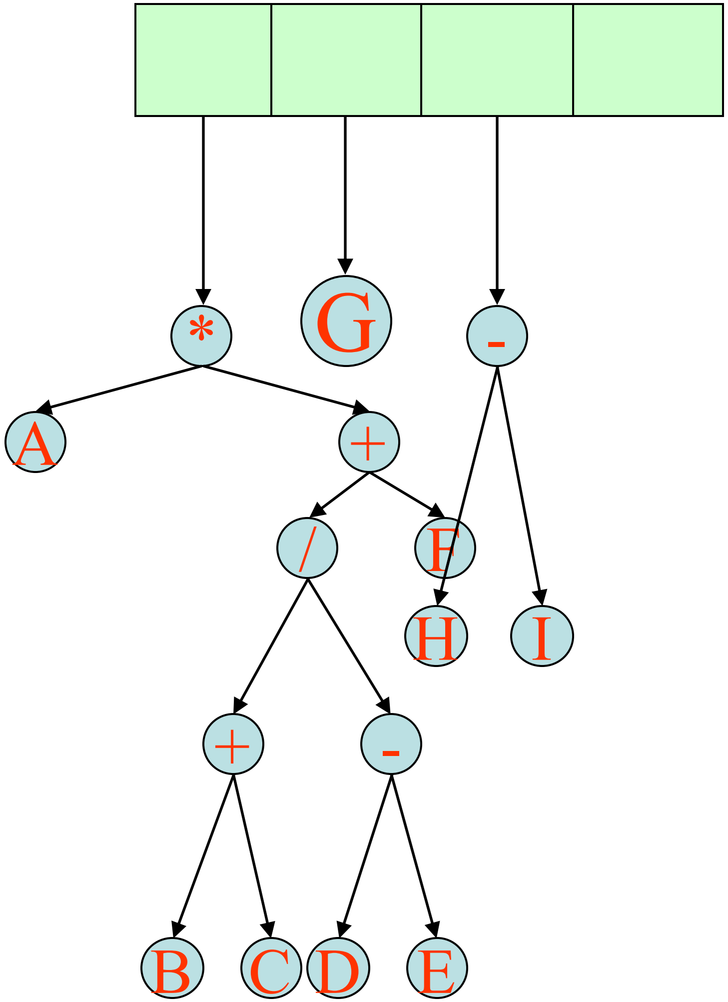
ABC+DE-/F+*GHI-/-

Example to Construction of Expression Trees - 14



ABC+DE-/F+*GHI-/-

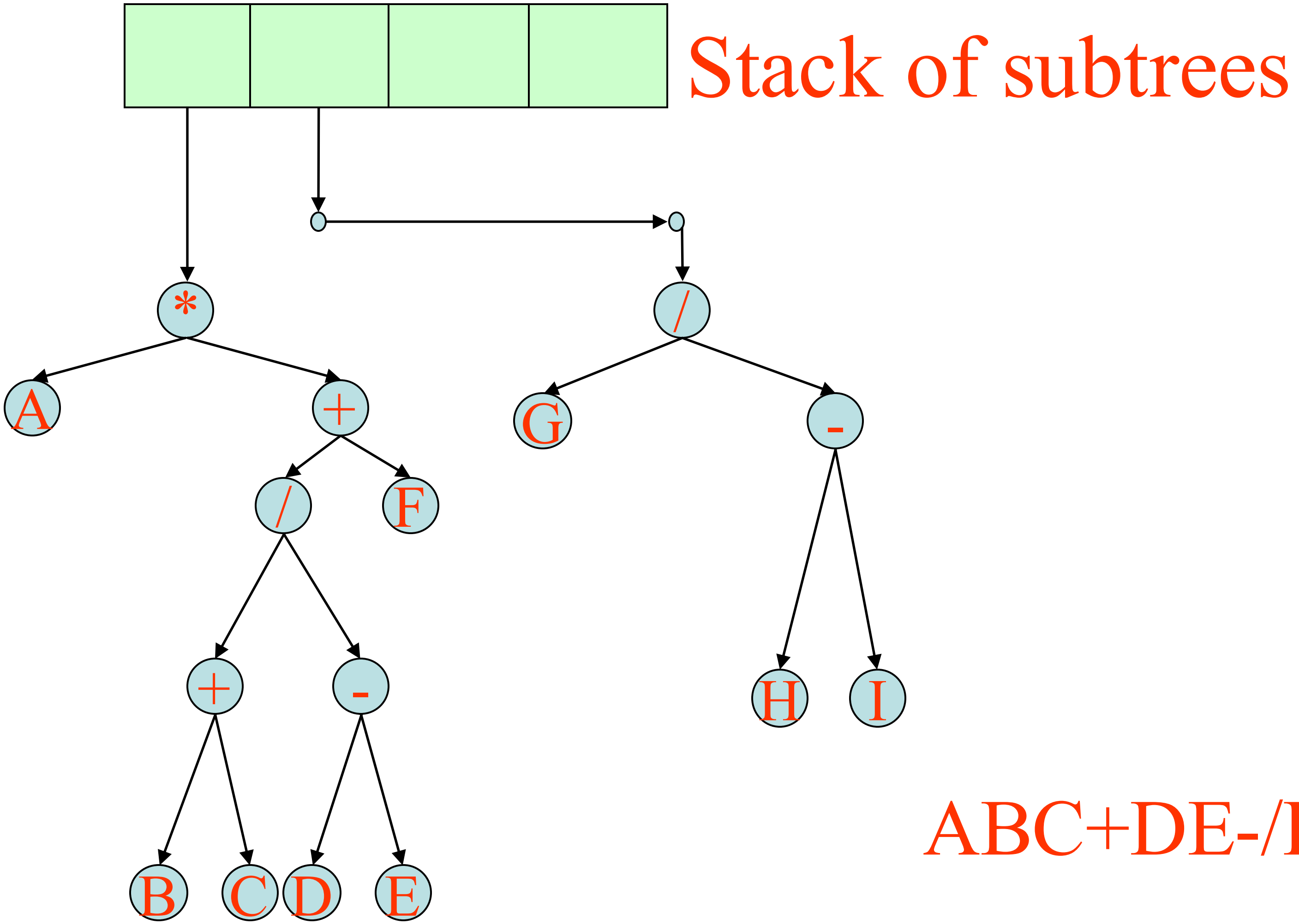
Example to Construction of Expression Trees - 15



Stack of subtrees

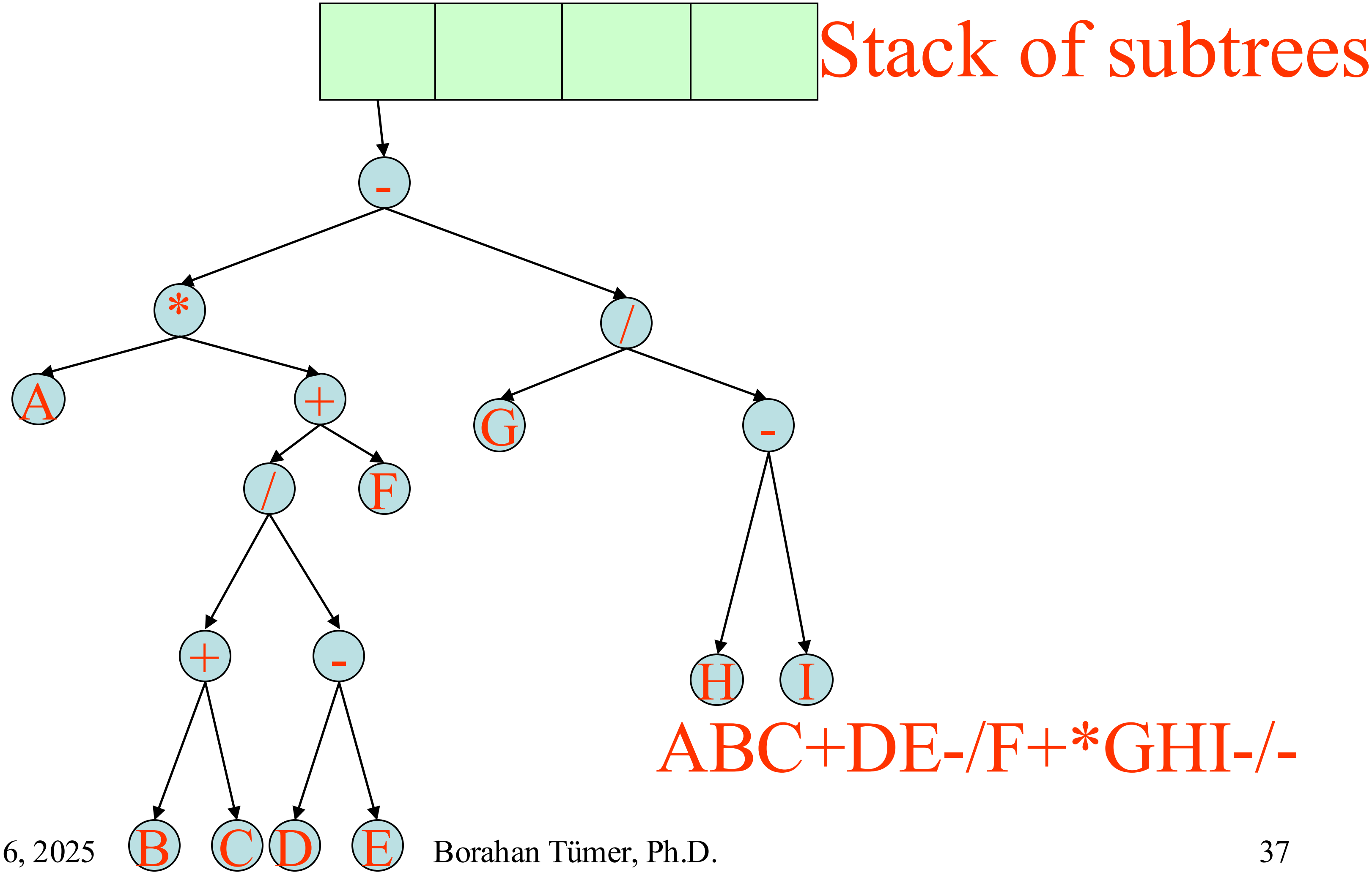
ABC+DE-/F+*GHI-/-

Example to Construction of Expression Trees - 16



ABC+DE-/F+*GHI-/-

Example to Construction of Expression Trees - 17



ABC+DE-/F+*GHI-/-

Binary Search Trees (BSTs)

BSTs are binary trees with keys placed in each node in such a manner that

the key of a node is

greater than all keys in its left sub-tree

and

***less than all keys in its right sub-tree*.**

Here, we assume *no replication of keys*. For replicating keys, the relations are modified as "*greater than or equal to*" or "*less than or equal to*" depending on the application.

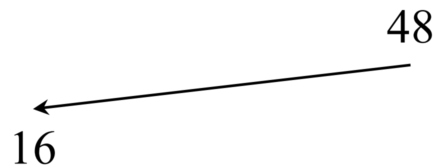
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30

48

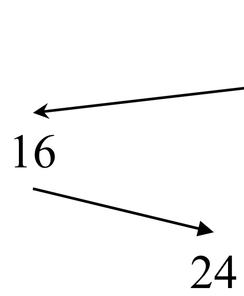
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



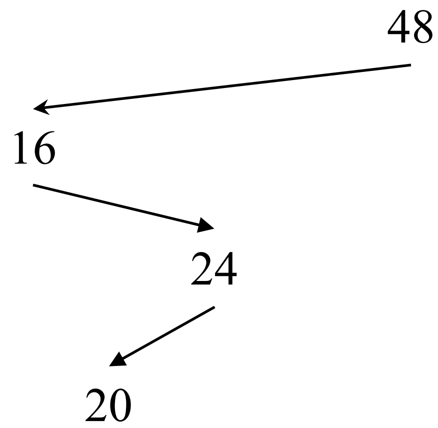
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



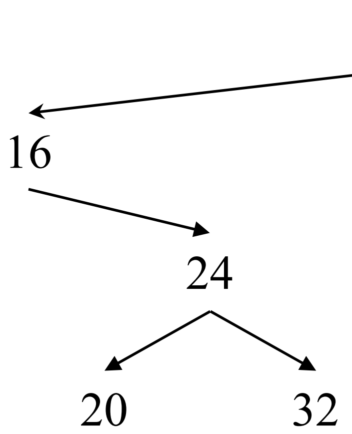
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



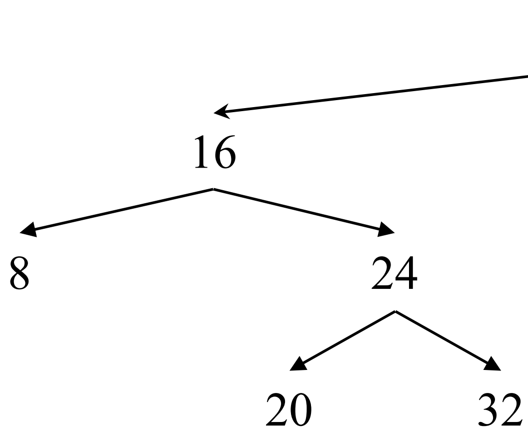
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



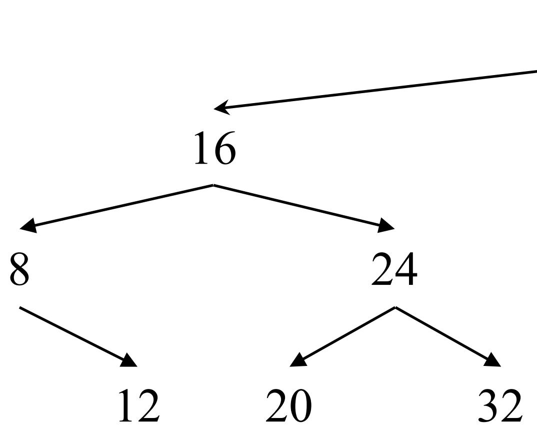
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



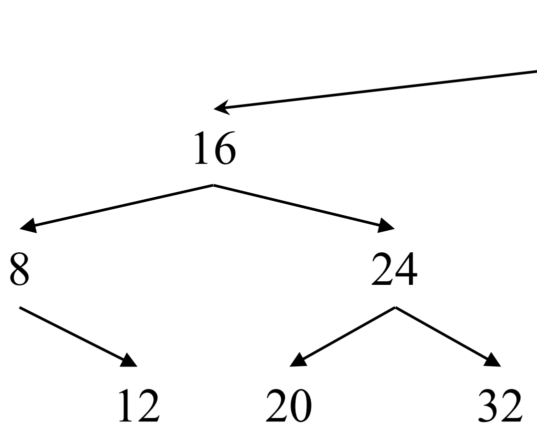
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



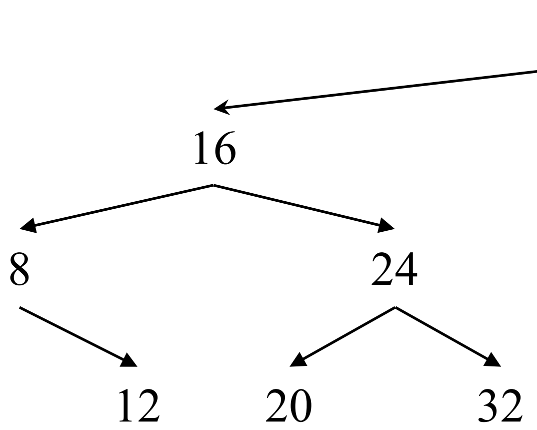
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



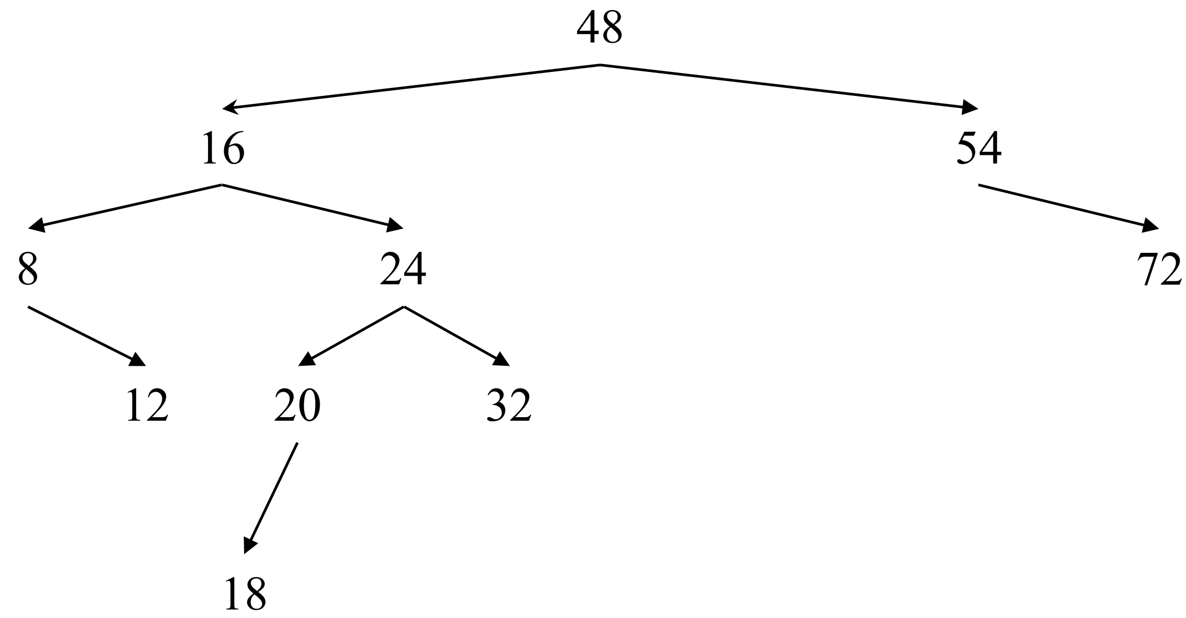
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



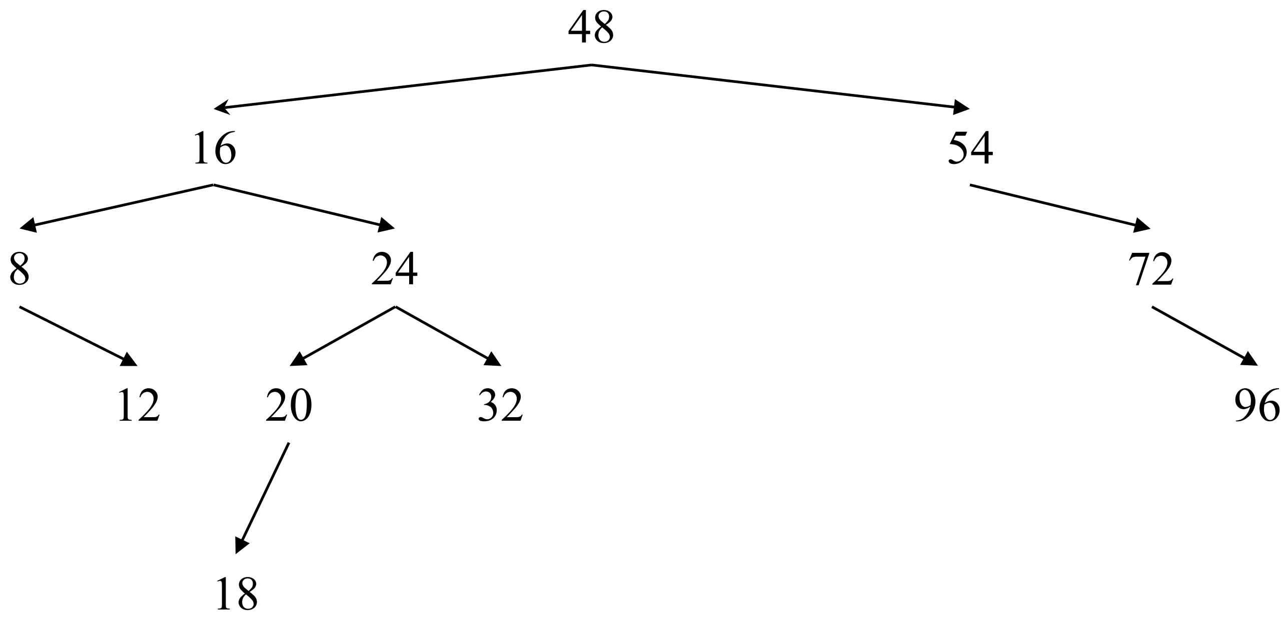
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



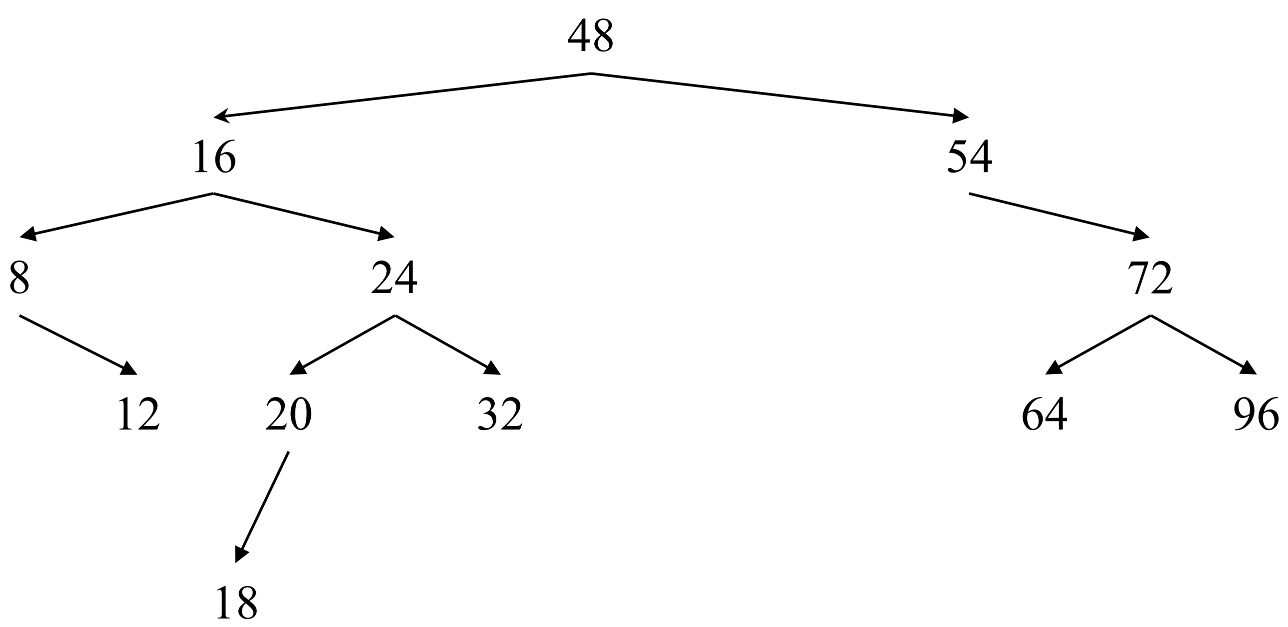
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



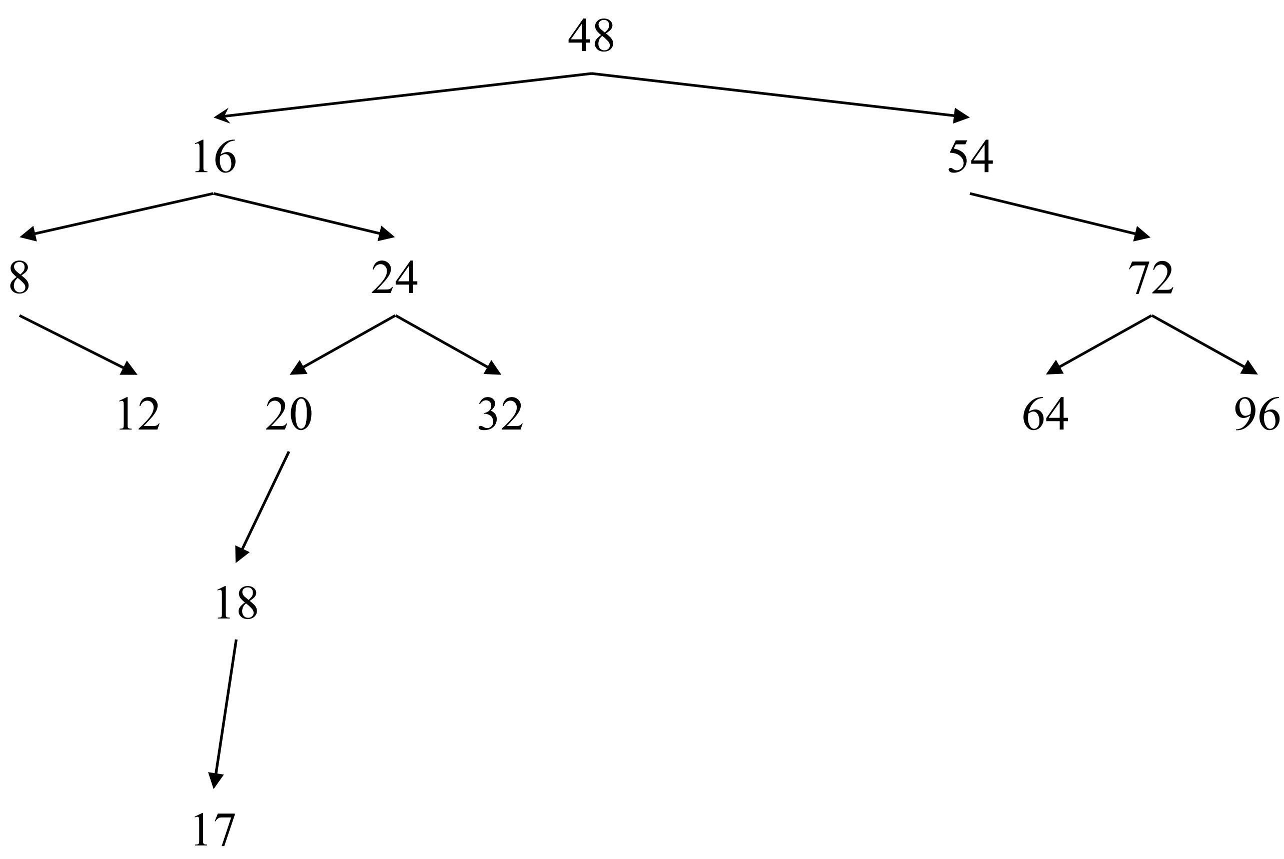
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



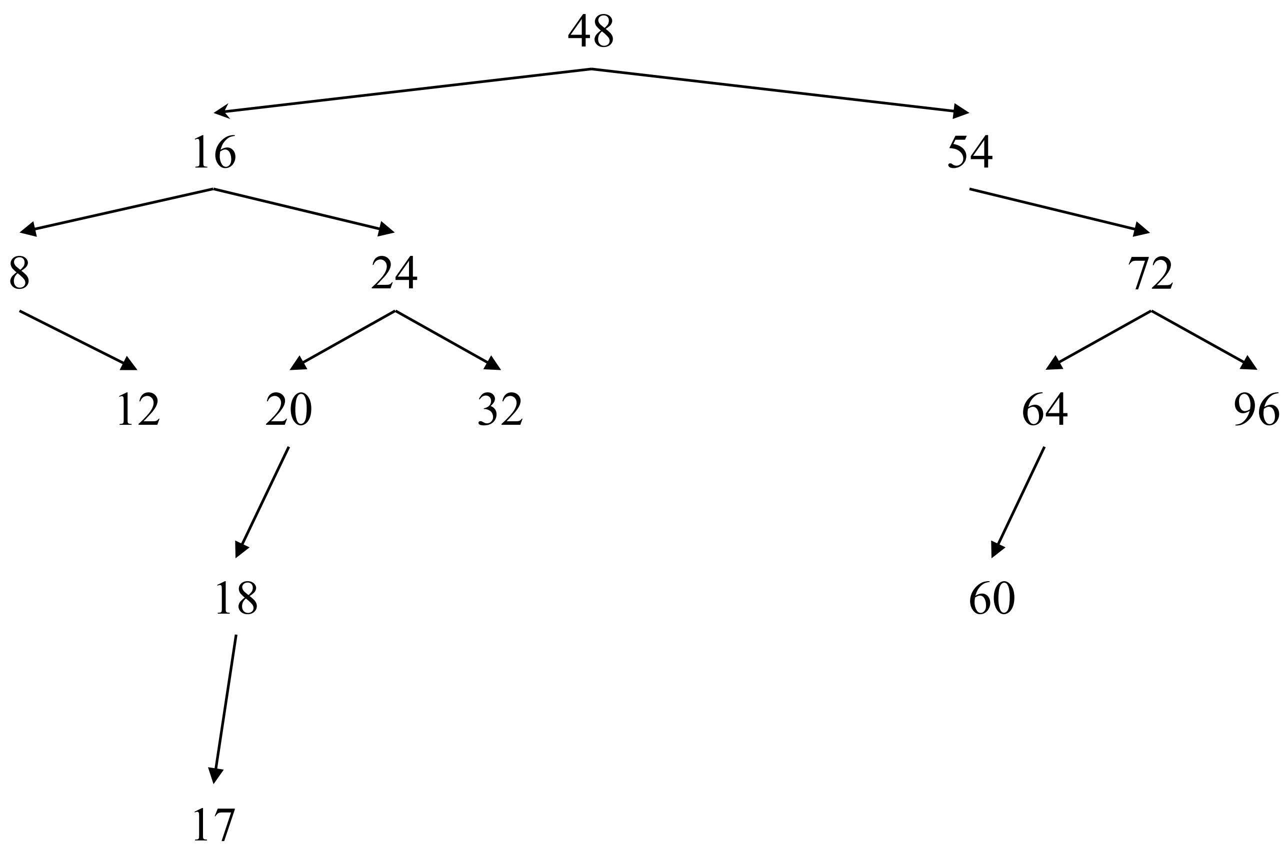
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



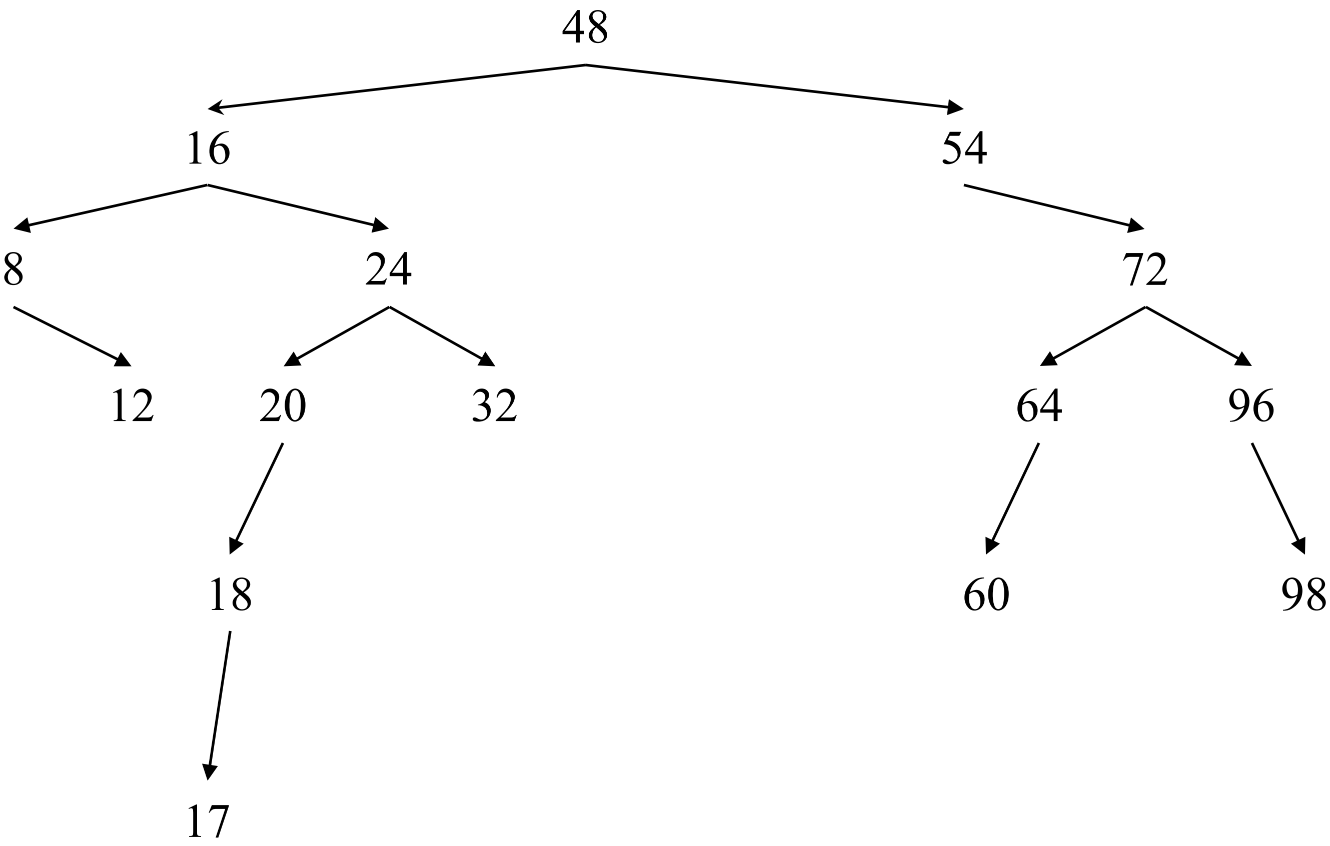
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



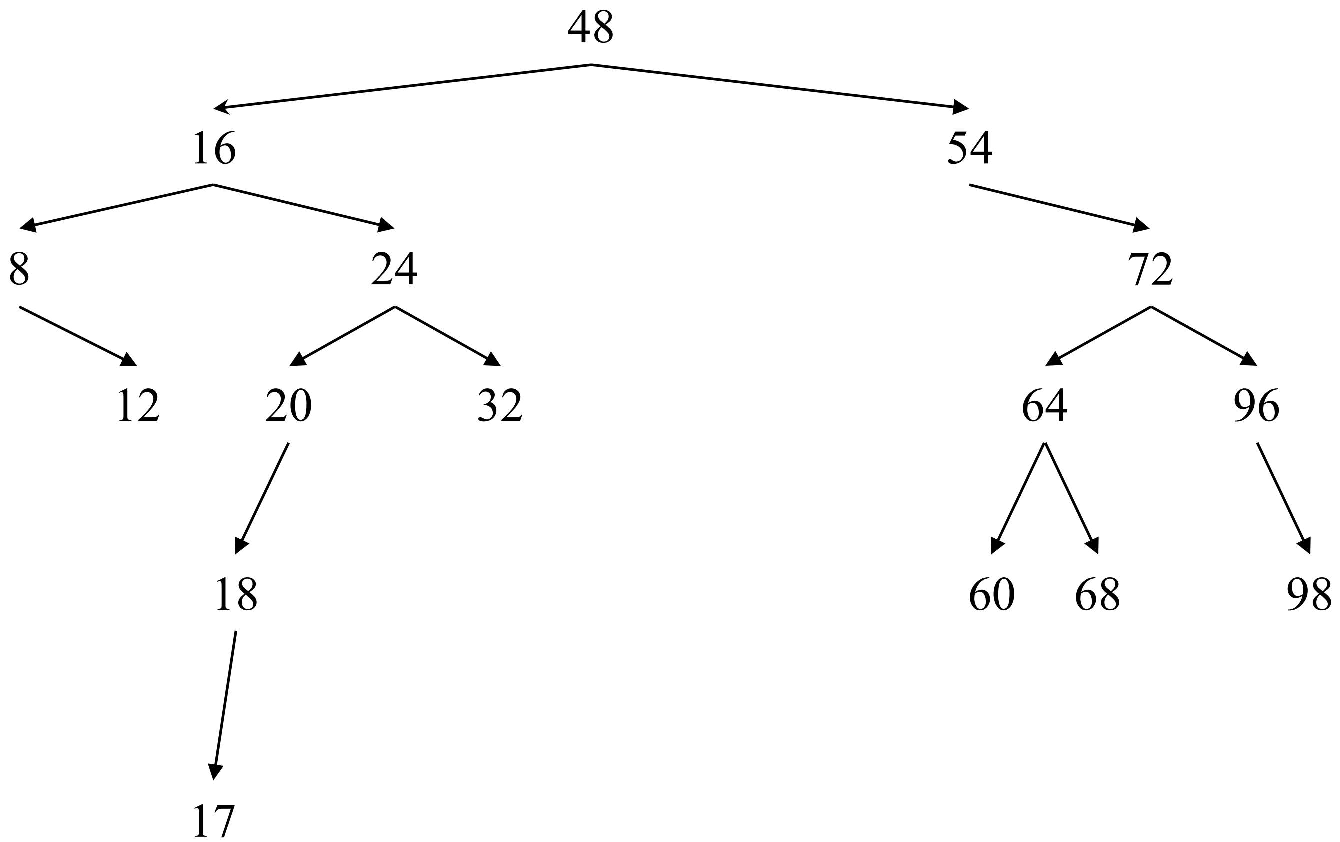
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



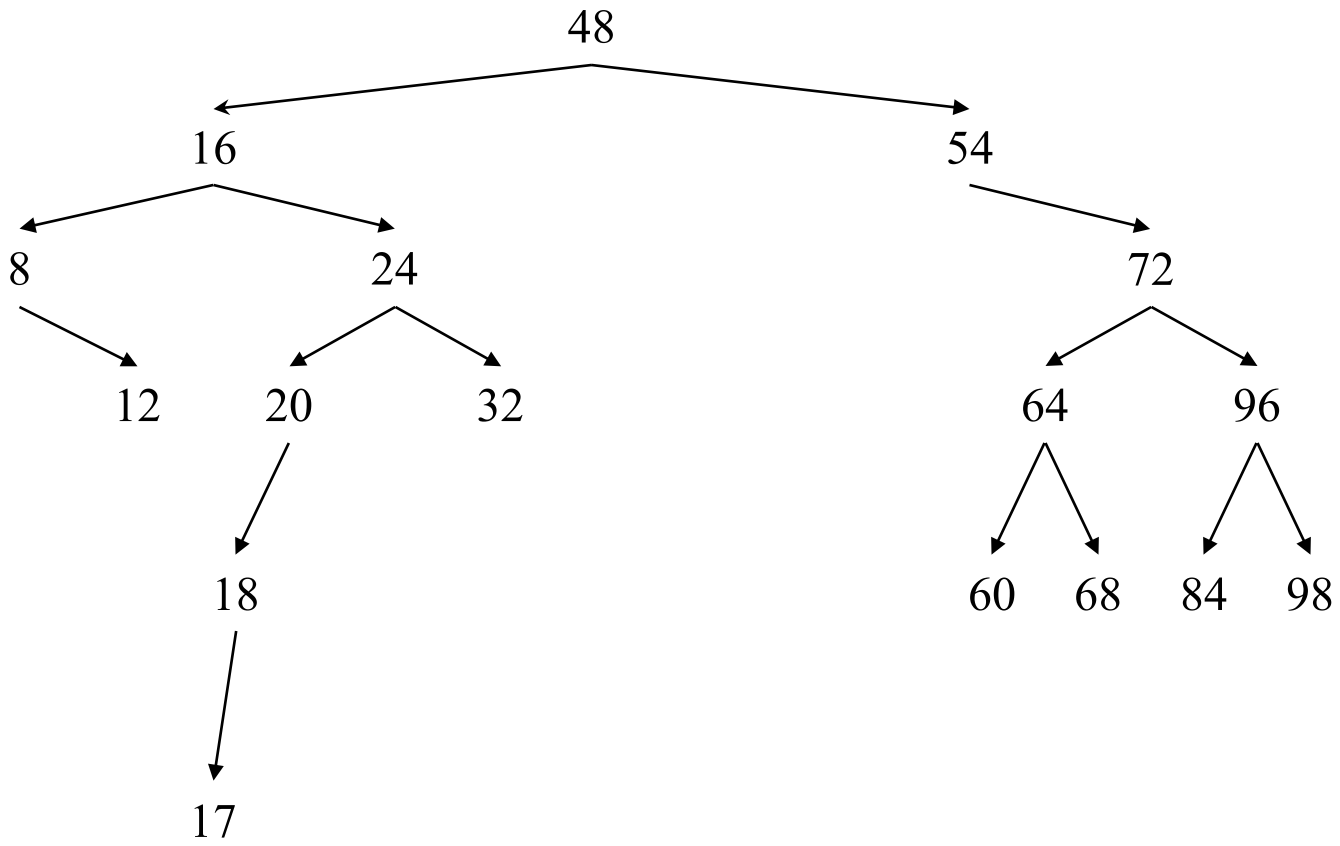
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



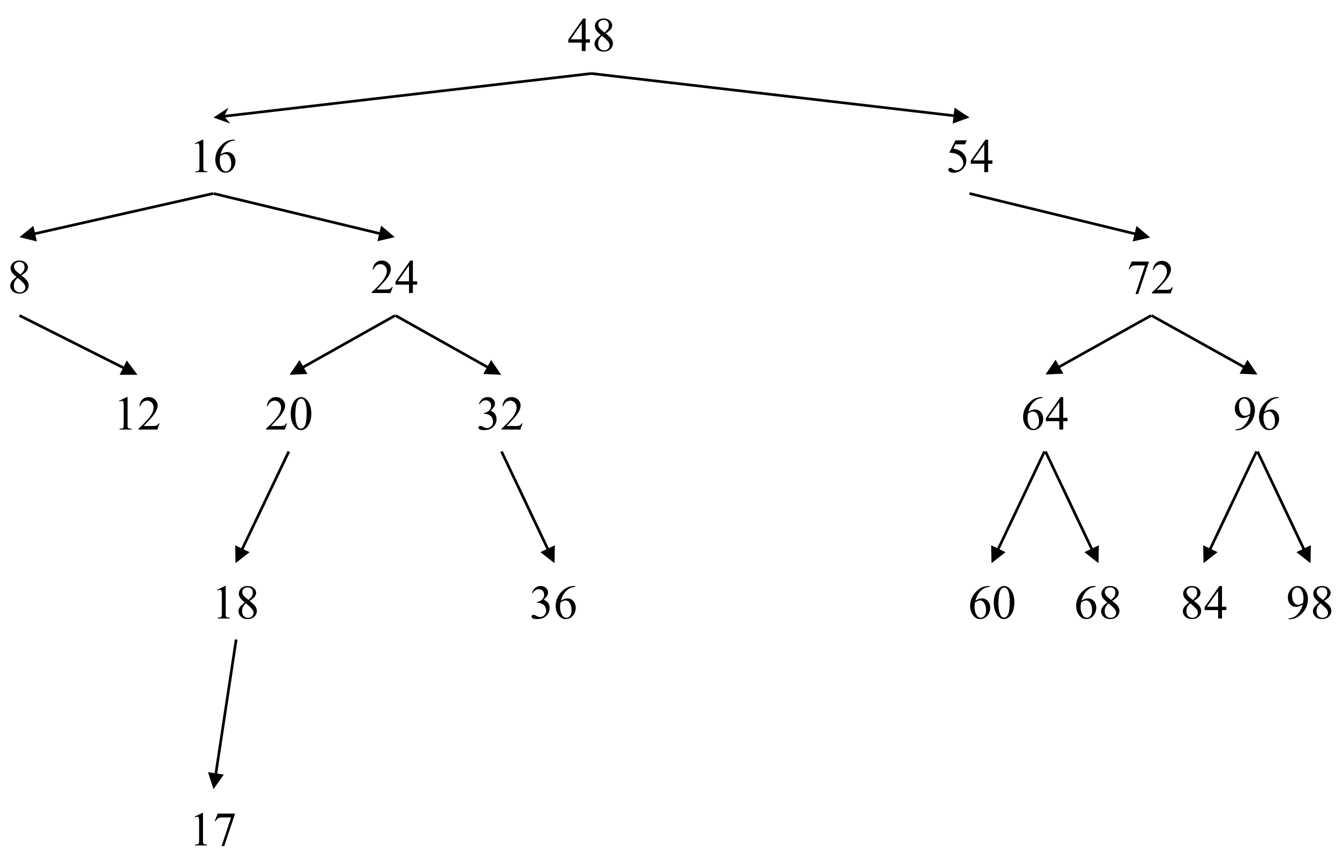
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



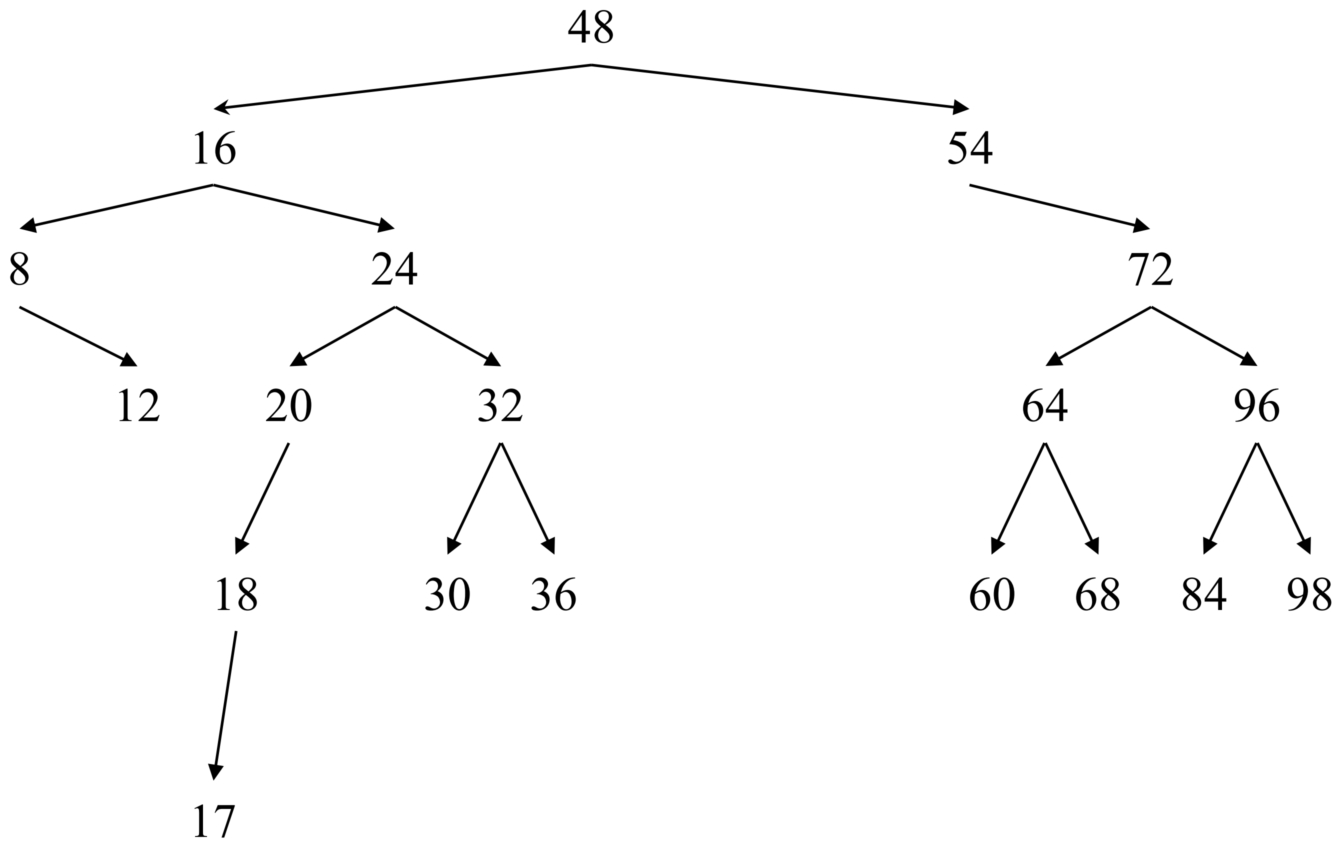
A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



A Binary Search Tree Example

48	16	24	20	32	8	12	54	72	18	96	64	17	60	98	68	84	36	30



Recursive Find Function in BSTs

```
int find (BTNodeType *p, int key)
{ // we assume infotype in BTNodeType is int
  if (p != NULL)
     if (key < p->data) find(p->left, key);
     else if (key > p->data) find(p->right, key);
          else if (p->data == key) printf (p->data);
}
```

Analysis of Find - 1

We would like to calculate the ***search time***, *t(n)*, it takes to find a given key in a BST with *n* nodes (or to find it is not in the BST!). First let us answer the following question:

Question: What is the *unit operation* performed to accomplish the search?

Answer: The *comparison* of the key searched with the key stored in the tree node (i.e., if (key < p->data) or if (key > p->data) or if (key == p->data)). At least one, at most three comparisons are made. key < p->data is always checked once. Hence we may assign 1 time unit to this condition.

Hence, what we need to do to find *t(n)* is to ***count, as a function of n***, *how many times the comparison* (i.e. condition 1) *is executed*.

Analysis of Find - 2

Now, we have converted the problem into one as in the following:

what is the <u>average number of comparisons (i.e., condition checks) performed</u> to find a given key in a BST?

or

<u>how many nodes, on average, are visited</u> to find a given key or to find it is not in the BST?

or, in a higher level of detail,

what is the <u>***average depth*** of the node</u> that stores the key we are looking for or, in case the key is not in the BST, what is the average depth of the leaf at the end of the path our search follows through?

Analysis of Find – 2: Average Depth

Average Depth (AD): denoted by d_{ave}

$$d_{ave} = \frac{total\ depth}{number\ of\ nodes} = \frac{t(n)}{n} = \frac{\sum_{i=1}^{n} dl_i}{n};$$

where dl_i denotes above the depth level (DL) the i^{th} node is at. *e.g.*, root (node #1)is at the 0^{th} DL ($dl_{i=1} = 0$), and the root's index is $i = 1$, *i.e.*, ($dl_{i=1}$)=*0.*

("e.g." and "i.e." stand for "for example" and "that is to say," respectively.)

Analysis of Find – 2: Average Depth

(!!!) Attention (!!!)

A BST or, in general, an ordered tree with ***k*** *DLs* has *DL indices* between ***0* and *k-1***.

Analysis of Find – 2: Internal Path Length

Internal path length (IPL) or total depth (TD): Another concept closely related to *total # of comparisons.*

Definition:

IPL is the sum of the length of all paths that connect the nodes of a tree.

TD is the sum of depths of all nodes in the binary tree topology.

IPL *f(n)*, AD *t(n)* and their relation: A 14-node BST example

1node 1comparison;

2 nodes 2 comp.'s each;

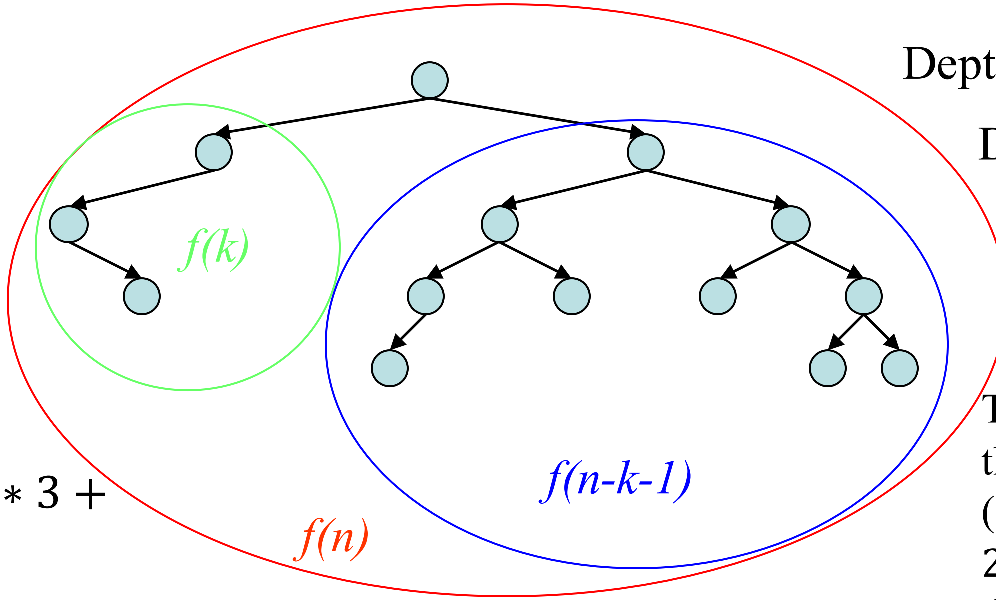
3 nodes 3 comp.'s each;

5 nodes 4 comp.'s each;

3 nodes 5 comp.'s each;

+

$t(n) = 1*1 + 2*2 + 3*3 + 5*4 + 3*5 = 49$



Depth Level # 0 or DL #0

DL #1; $i = 2,3$

DL #2; $i = 4,6,7$

DL #3; $i = 9,12,\ldots,15$

DL #4; $i = 24,30,31$

The node index i is so that each j^{th} node's (i.e., $i = j$) children are $2j$ and $2j + 1$; you see that missing nodes' indices are not used above.

But in the function $f(n)$ at left: $n = 14$, $k \in \{1, \ldots, n\}$

$$f(14) = f(3) + f(10) + 14 - 1$$
$$f(14) = 3 + 19 + 14 - 1$$
$$f(14) = 35$$

$$\boldsymbol{f(n) = f(k) + f(n-k-1) + n - 1}$$

Relation among IPL (i.e., total depth) and # of comparisons : $\boldsymbol{t(n) = f(n) + n}$

$\boldsymbol{t(n) = f(n) + n} \Rightarrow t(n) = 35 + 14 = 49$

Analysis of Find - 3

Now, we turn back to formulating *t(n)*:

We will do the formulation for three cases:

1. Average case
2. Worst case
3. Best case

Analysis of Find: Average Case - 1

Average case ⇔ (i.e., *BST has any number of depth levels ranging min. to max.*)

- Each node has a *left and right subtree*.
- The search time consists of the search time of the left subtree, right subtree and the time spent for the root in the main BST.
- We will
  1. formulate and find the *internal path length (or the total depth) f(n)* of a BST with *n* nodes
  2. then compute the total # of comparisons *t(n)* and average # of comparisons using *f(n)*.

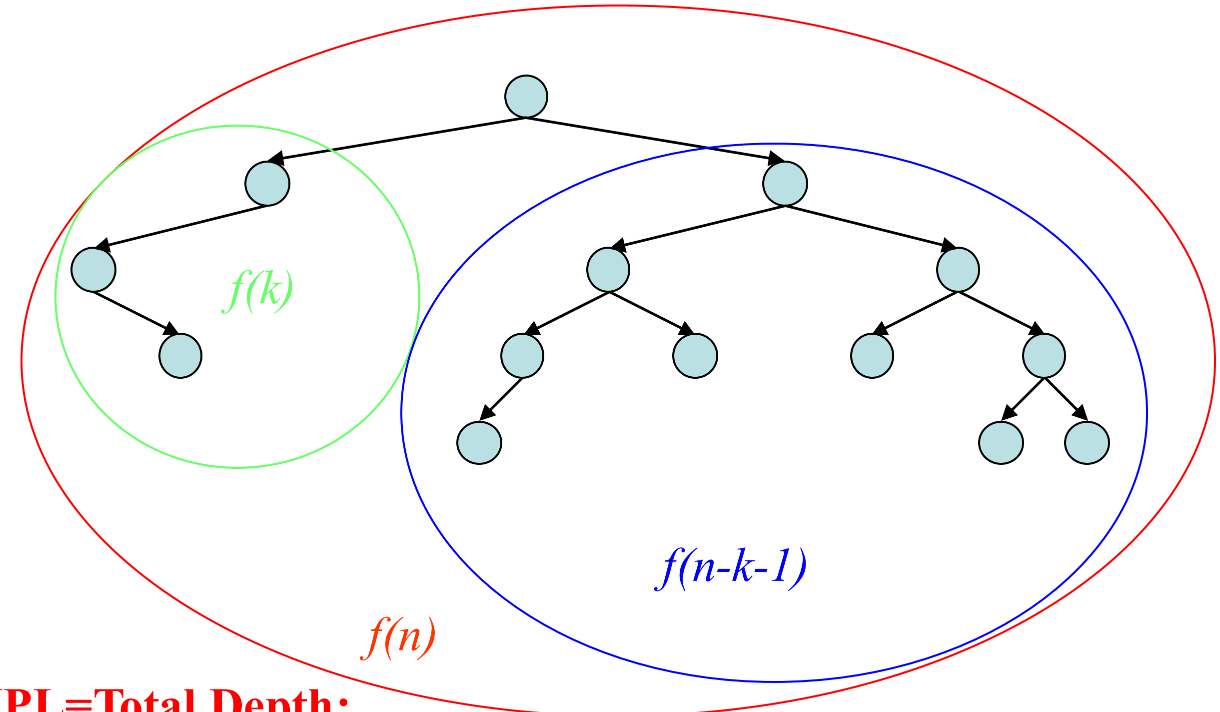
Average Case – Formulating IPL

- Suppose an *n*-node BST.
- Its left and right subtrees have have *k* and *n-k-1* nodes, respectively. ($n = n_{Left} + n_{Right} + 1$; *1 is added for the root*).
- Then the IPL may be expressed as follows:

$$f(n) = f(k) + f(n - k - 1) + n - 1$$

where the term "*n-1*" is added to the sum of depths by the fact that all nodes in the BST are one level deeper in the main BST (because of the root of the main BST).

Average Case: Example tree with *f(14)*



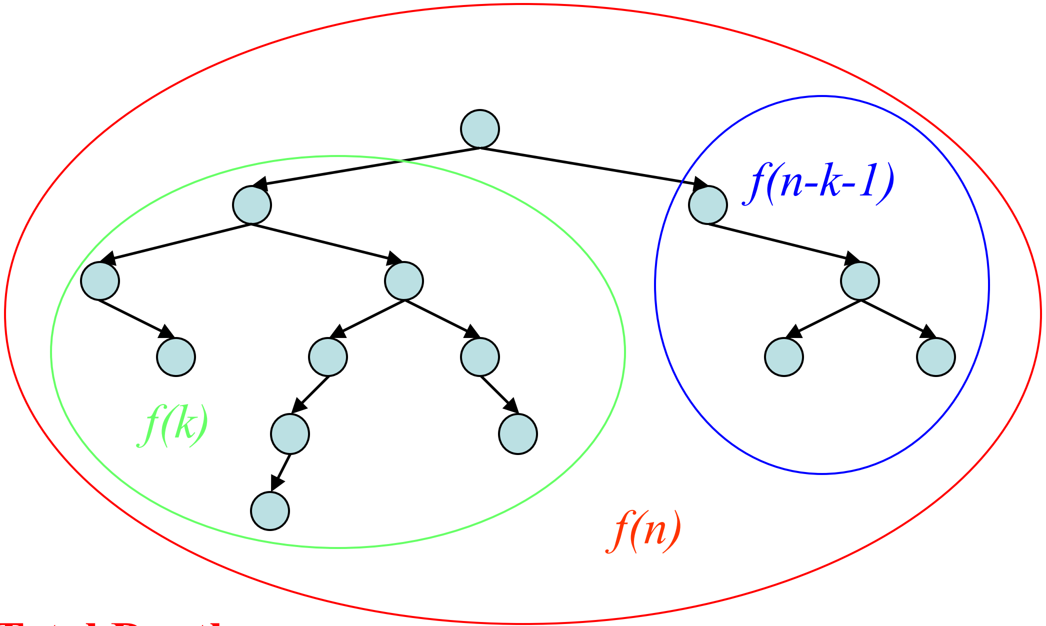
IPL=Total Depth:

$$f(14) = f(3) + f(10) + 14 - 1$$
$$f(14) = 3 + 19 + 14 - 1$$
$$f(14) = 35$$

Total # of comparisons:

$$t(n) = f(n) + n \Rightarrow t(n) = 35 + 14 = 49$$

Average Case: Another Example



IPL =Total Depth:

$$f(14) = f(9) + f(4) + 14 - 1$$
$$f(14) = 18 + 5 + 14 - 1$$
$$f(14) = 36\ (IPL)$$

Total # of comparisons :

$$t(n) = f(n) + n \Rightarrow t(n) = 36 + 14 = 50$$

Average Case - 2

$f(n) = f(k) + f(n-k-1) + n - 1;$

Since we consider any possible BST topology in the "average case" *k* and *n-k-1* can be any number between 0 and *n-1*. To come up with a general solution that satisfies the above condition, we can average *f(k)* and *f(n-k-1)*:

$$f_{ave}(n) = \left\{ \frac{2}{n} \sum_{k=0}^{n-1} f(k) \right\} + n - 1$$

From this point on we will drop the subscript off and use *f(n)* to denote $f_{ave}(n)$.

Average Case - 3

$$f(n) = \frac{2}{n}\sum_{k=0}^{n-1} f(k) + n - 1 \Big| * n$$

$$nf(n) = 2\sum_{k=0}^{n-1} f(k) + n(n-1); \qquad (I)$$

$$(n-1)f(n-1) = 2\sum_{k=0}^{n-2} f(k) + (n-1)(n-2) \qquad (II)$$

$$nf(n) - (n-1)f(n-1) = 2f(n-1) + 2(n-1); (I) - (II)$$

$$nf(n) = (n+1)f(n-1) + 2(n-1);$$

$$\frac{f(n)}{n+1} = \frac{f(n-1)}{n} + 2\frac{(n-1)}{n(n+1)}$$

Average Case - 4

$$\frac{f(n)}{n+1} = \frac{f(n-1)}{n} + 2\frac{n-1}{n(n+1)}$$

$$\frac{f(n-1)}{n} = \frac{f(n-2)}{n-1} + 2\frac{n-2}{(n-1)n}$$

$$\vdots$$

$$\frac{f(2)}{3} = \frac{f(1)}{2} + \frac{1}{2*3}$$

$$\frac{f(1)}{2} = \frac{f(0)}{1} + 0$$

$$\frac{f(n)}{n+1} = f(0) + 2\sum_{i=1}^{n}\frac{i-1}{i(i+1)}$$

$$f(n) = (n+1)f(0) + 2(n+1)\sum_{i=1}^{n}\frac{i-1}{i(i+1)}$$

Average Case - 5

$$f(n) = (n+1)f(0) + 2(n+1)\sum_{i=1}^{n}\frac{i-1}{i(i+1)};\ \frac{i-1}{i(i+1)} = \frac{2}{i+1} - \frac{1}{i}; f\,(0) = 0$$

$$f(n) = 2(n+1)\left[2\sum_{i=1}^{n}\frac{1}{i+1} - \sum_{i=1}^{n}\frac{1}{i}\right];\ \sum_{i=1}^{n}\frac{1}{i+1} = \sum_{2}^{n}\frac{1}{i} + \frac{1}{n+1}$$

$$f(n) = 2(n+1)\left[\sum_{i=2}^{n}\frac{1}{i} - 1 + \frac{2}{n+1}\right]$$

$$f(n) \cong 4 + 2(n+1)[\log(n) - 1];\sum_{i=2}^{n}\frac{1}{i} = O(\log(n))\ since \int\frac{dx}{x} = a\log x + c$$

$$f(n) = O(n + n\log(n)) = O(n\log(n))$$

To find the average (per node) number of comparisons, ***N_{ave}***, we divide *f(n)* by the number of nodes *n*: ***$N_{ave} = f(n)/n \Rightarrow O(\log n)$***

Analysis of Find: Worst Case - 1

Worst Case BST: (i.e., *BST has a maximum number of depth levels*)

The worst case BST is one with the deepest path. An n-node such BST is one with n depth levels.

Question: How many such n-node BSTs are there?

We will formulate and find the *internal path length f(n)* in a worst case n-node BST.

In such a BST one subtree of the root forms a linked list of $n-1$ nodes whereas the other subtree has no nodes. Then the *depth* may be expressed as follows:

$$f(n)=f(n-1)+n-1.$$

Worst Case - 2

$$f(n) = f(n-1) + n - 1; f(1) = 0$$

$$CE: \quad (x-1)(x-1)^2 = 0$$

$$f(n) = c_1 + c_2 n + c_3 n^2$$

$$f(1) = c_1 + c_2 + c_3 = 0$$

$$f(2) = c_1 + c_2 2 + c_3 4 = 1$$

$$f(3) = c_1 + c_2 3 + c_3 9 = 3$$

$$\vdots$$

$$f(n) \in O(?)$$

Order of $f(n)$: $f(n)=O(n^2)$

Analysis of Find: Best Case - 1

Best Case BST: (i.e., *BST has a minimum number of depth levels*)

The best case BST is one with the least tree depth. An *n*-node such BST is one with $\boxed{\lfloor \log_2 n \rfloor + 1}$ depth levels.

Question: How many such *n*-node BSTs are there?

We will formulate and find the *internal path length f(n)* in a best case *n*-node BST.

In such a BST both subtrees of the root have the same number of nodes (i.e., n/2 nodes). Then the *depth* may be expressed as follows:

$$f(n)=2f(n/2)+n-1. \text{ (conditional recurrence!!!)}$$

Best Case - 2

$$f(n) = 2f(n/2) + n - 1; f(1) = 0; n = 2^k$$

$$f(k) = 2f(k-1) + 2^k - 1$$

$$CE: \quad (x-2)(x-2)\ (x-1) = 0$$

$$f(k) = c_1 2^k + c_2 k 2^k$$

$$f(n) = c_1 n + c_2 n \log_2(n)$$

$$f(1) = c_1 = 0$$

$$f(2) = 2c_1 + 2c_2 = 1 \Rightarrow c_2 = \tfrac{1}{2}$$

$$f(n) = \frac{1}{2} n \log_2(n)$$

$$f(n) \in O(?)$$

Order of *f(n)*: *f(n)=O(nlgn)*

Non-recursive FindMin Function in BSTs

```
BTNodeType *findMin (BTNodeType *p)
{// returns a pointer to node with the minimum key in the BST.
 // which one is the minimum key in a BST?

  if (p != NULL)
        while (p->left != NULL) p=p->left;
  return p;

}
```

Recursive FindMax Function in BSTs

```
BTNodeType *findMax (BTNodeType *p)
{// returns a pointer to node with the maximum key in the BST
 // which one is the maximum key in a BST?
   if (p == NULL) return NULL;
   if (p->right == NULL) return p;
   return findMax(p->right);
}
```

Recursive Insert Function in BSTs

```
BTNodeType *insert (infotype * key, BTNodeType *p)
{// returns a pointer to new node with key inserted in the BST.
    if (p == NULL) {                                    //empty tree
        p=malloc(sizeof(BTNodeType));
        if (p == NULL) return OutofMemoryError;         //no heap space left!!!
        p->data=key;
        p->left=p->right=NULL;
        return p;
    }
    else if (key < p->data)                             //tree exists, key < root
         p->left=insert(key,p->left);
    else if (key > p->data)                             //tree exists, key > root
         p->right=insert(key,p->right);
    return p;                                           //key found
}
```

Recursive Remove Function in BSTs

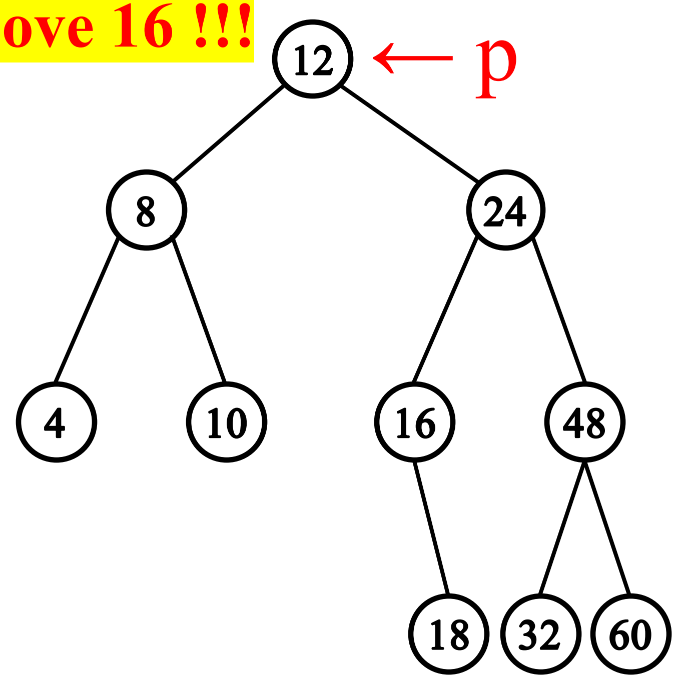
```
BTNodeType *Remove (infotype *key, BTNodeType *p)
{// returns a pointer to node replacing the node removed.
       BTNodeType *TempPtr;
       if (p == NULL)                                      //empty tree
              return errorMessage("empty tree");
       if (key < p->data)
         p->left=Remove(key,p->left);
       else if (key > p->data)
         p->right=Remove(key,p->right);
       else                                                //node with key found!!!
         if (p->right && p->left) {                        //if node has two children
            TmpPtr=findMin(p->right);                      //find smallest key of right subtree of node with key
            p->data=TmpPtr->data;                          //replace removed data by smallest key of right subtree
            p->right=Remove(p->data,p->right);
       }
       else {                                              //if node has less than two children
            TmpPtr=p;
            if (p->left == NULL) p=p->right;
            else if (p->right == NULL) p=p->left;
            free(TmpPtr);
       }
       return p;
}
```

Removing a key at a node with a single child

An Example…

Removing a key of node with a single child

Remove 16 !!!




```
BTNodeType *Remove (infotype *key, BTNodeType *p)
{// returns a pointer to node replacing the node removed.
      BTNodeType *TempPtr;
      if (p == NULL)
              return errorMessage("empty tree");
      if (key < p->data)
        p->left=Remove(key,p->left);
      else if (key > p->data)
        p->right=Remove(key,p->right);
      else
        if (p->right && p->left) {
           TmpPtr=findMin(p->right);
           p->data=TmpPtr->data;
           p->right=Remove(p->data,p->right);
      }
      else {
           TmpPtr=p;
           if (p->left == NULL) p=p->right;
           else if (p->right == NULL) p=p->left;
           free(TmpPtr);
      }
      return p;
```

Call #0: non-recursive

p points to root! i.e., p → 12

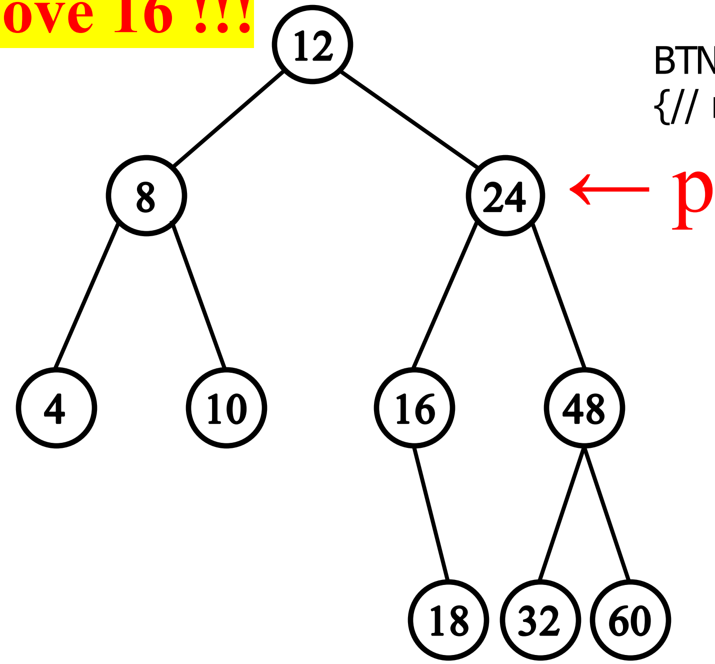
key > p→data

Red background: condition false

Green backgrnd: condition true

Removing a key of node with a single child

Remove 16 !!!




```
BTNodeType *Remove (infotype *key, BTNodeType *p)
{// returns a pointer to node replacing the node removed.
    BTNodeType *TempPtr;
    if (p == NULL)
            return errorMessage("empty tree");
    if (key < p->data)
      p->left=Remove(key,p->left);
    else if (key > p->data)
      p->right=Remove(key,p->right); Call #1 point !!!
    else
      if (p->right && p->left) {
         TmpPtr=findMin(p->right);
         p->data=TmpPtr->data;
         p->right=Remove(p->data,p->right);
    }
    else {
         TmpPtr=p;
         if (p->left == NULL) p=p->right;
         else if (p->right == NULL) p=p->left;
         free(TmpPtr);
    }
    return p;
```

Call #1: recursive

p points to root! i.e., p → 24

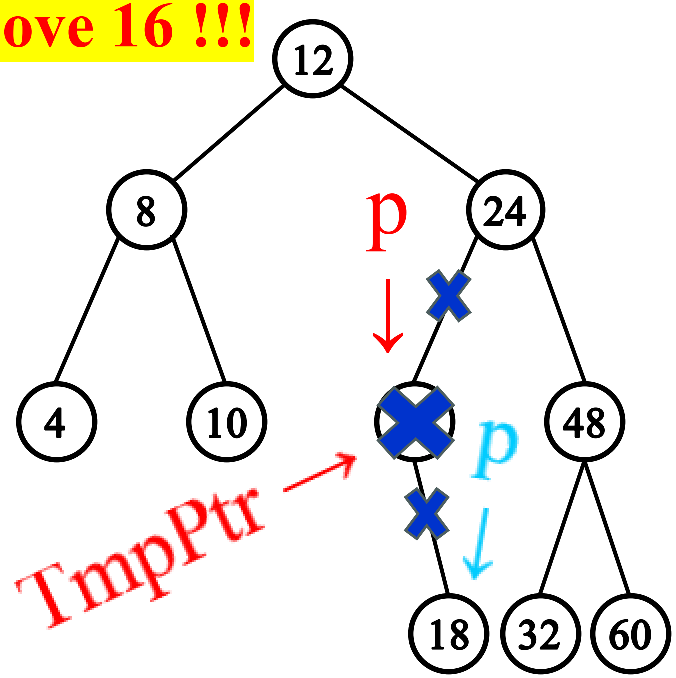
key < p→data

Red background: condition false

Green backgrnd: condition true

Removing a key of node with a single child

Remove 16 !!!




```
BTNodeType *Remove (infotype *key, BTNodeType *p)
{// returns a pointer to node replacing the node removed.
      BTNodeType *TempPtr;
      if (p == NULL)
                return errorMessage("empty tree");
      if (key < p->data)
        p->left=Remove(key,p->left); Call #2 point !!!
      else if (key > p->data)
        p->right=Remove(key,p->right);
      else                                    //key = p→data
        if (p->right && p->left) {       //double child node
           TmpPtr=findMin(p->right);
           p->data=TmpPtr->data;
           p->right=Remove(p->data,p->right);
      }
      else {                         //single child node
           TmpPtr=p;
           if (p->left == NULL) p=p->right;
           else if (p->right == NULL) p=p->left;
           free(TmpPtr);
      }
      return p;                //p → 18 returned
```

Call #2: recursive

p points to root! i.e., p → 16

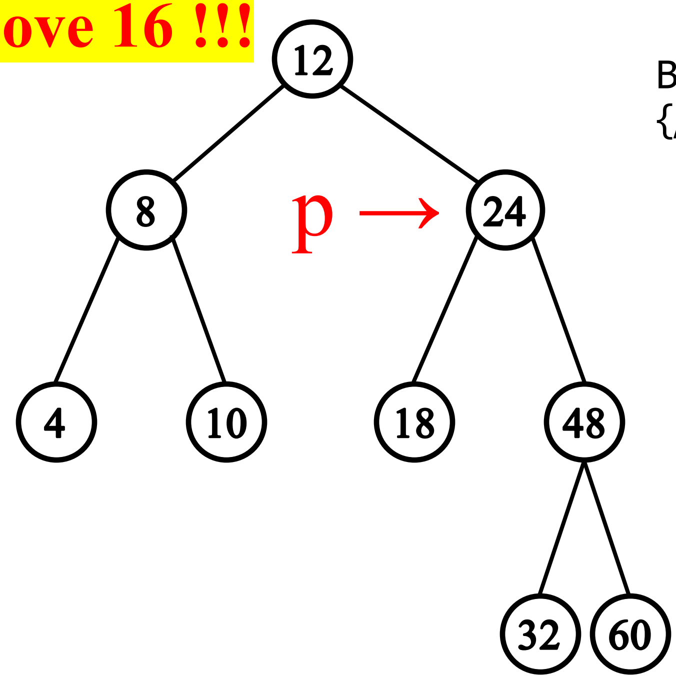
key = p→data

Red background: condition false

Green backgrnd: condition true

Removing a key of node with a single child

Remove 16 !!!



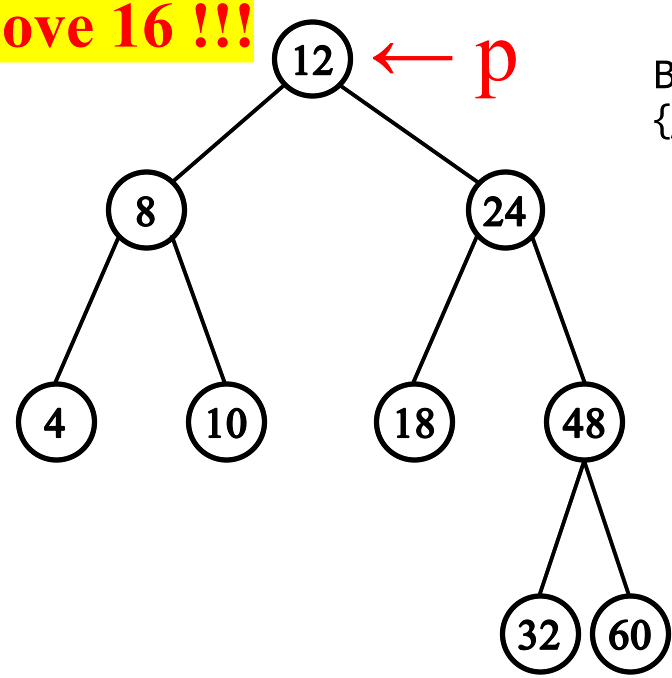

```
BTNodeType *Remove (infotype *key, BTNodeType *p)
{// returns a pointer to node replacing the node removed.
      BTNodeType *TempPtr;
      if (p == NULL)
            return errorMessage("empty tree");
      if (key < p->data)
        p->left=Remove(key,p->left; Call #2 point !!!
      else if (key > p->data)
        p->right=Remove(key,p->right);
      else
        if (p->right && p->left) {
          TmpPtr=findMin(p->right);
          p->data=TmpPtr->data;
          p->right=Remove(p->data,p->right);
      }
      else {                    //single child node
          TmpPtr=p;
          if (p->left == NULL) p=p->right;
          else if (p->right == NULL) p=p->left;
          free(TmpPtr);
      }
      return p;
```

Return from Call #2 back to Call #1 at the point of Call #2

Address of 18 returned to p→left with p pointing to 24.

Removing a key of node with a single child

Remove 16 !!!



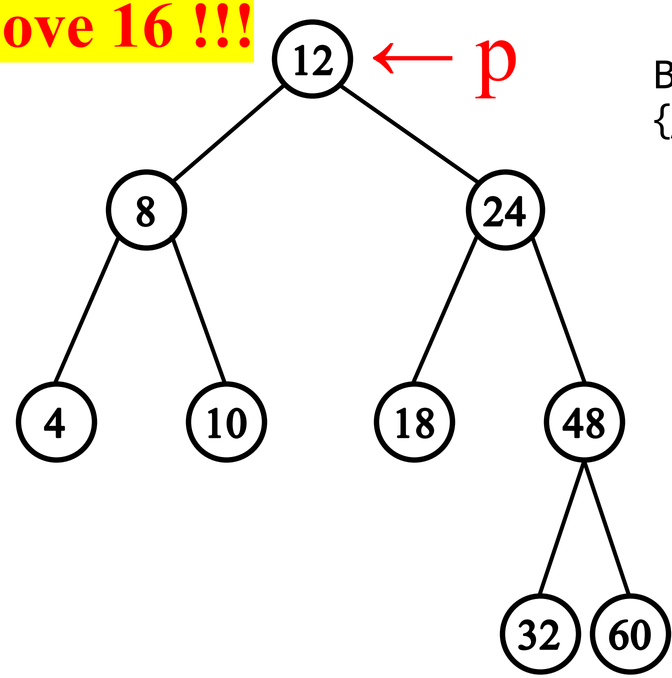

```
BTNodeType *Remove (infotype *key, BTNodeType *p)
{// returns a pointer to node replacing the node removed.
      BTNodeType *TempPtr;
      if (p == NULL)
            return errorMessage("empty tree");
      if (key < p->data)
        p->left=Remove(key,p->left);
      else if (key > p->data)
        p->right=Remove(key,p->right); Call #1 point !!!
      else
        if (p->right && p->left) {
           TmpPtr=findMin(p->right);
           p->data=TmpPtr->data;
           p->right=Remove(p->data,p->right);
      }
      else {
           TmpPtr=p;
           if (p->left == NULL) p=p->right;
           else if (p->right == NULL) p=p->left;
           free(TmpPtr);
      }
      return p;
```

Return from Call #1 back to Call #0 at the point of Call #1

Address of 24 returned to p→right with p pointing to 12.

Removing a key at node with a single child

Remove 16 !!!



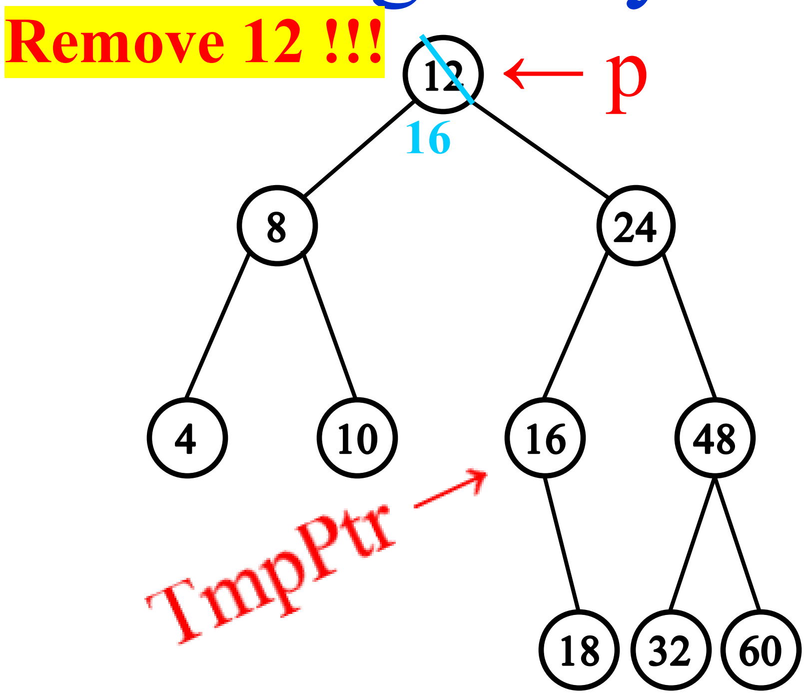

```
BTNodeType *Remove (infotype *key, BTNodeType *p)
{// returns a pointer to node replacing the node removed.
    BTNodeType *TempPtr;
    if (p == NULL)
            return errorMessage("empty tree");
    if (key < p->data)
      p->left=Remove(key,p->left);
    else if (key > p->data)
      p->right=Remove(key,p->right);
    else
      if (p->right && p->left) {
         TmpPtr=findMin(p->right);
         p->data=TmpPtr->data;
         p->right=Remove(p->data,p->right);
    }
    else {
         TmpPtr=p;
         if (p->left == NULL) p=p->right;
         else if (p->right == NULL) p=p->left;
         free(TmpPtr);
    }
    return p;
```

Return from Call #0
Address of 12 (i.e., modified BST) returned to the invoker function with 16 removed.

Removing a key at a node with two children

An Example…

Removing a key at a node with two children




```
BTNodeType *Remove (infotype *key, BTNodeType *p)
{// returns a pointer to node replacing the node removed.
    BTNodeType *TempPtr;
    if (p == NULL)
        return errorMessage("empty tree");
    if (key < p->data)
      p->left=Remove(key,p->left);
    else if (key > p->data)
      p->right=Remove(key,p->right);
    else                                  //key = p→data
      if (p->right && p->left) {          //double child node
        TmpPtr=findMin(p->right);
        p->data=TmpPtr->data;
        p->right=Remove(p->data,p->right);
    }
    else {
        TmpPtr=p;
        if (p->left == NULL) p=p->right;
        else if (p->right == NULL) p=p->left;
        free(TmpPtr);
    }
    return p;
```

Call #0: non-recursive

p points to root! i.e., p → 12

key = p→data

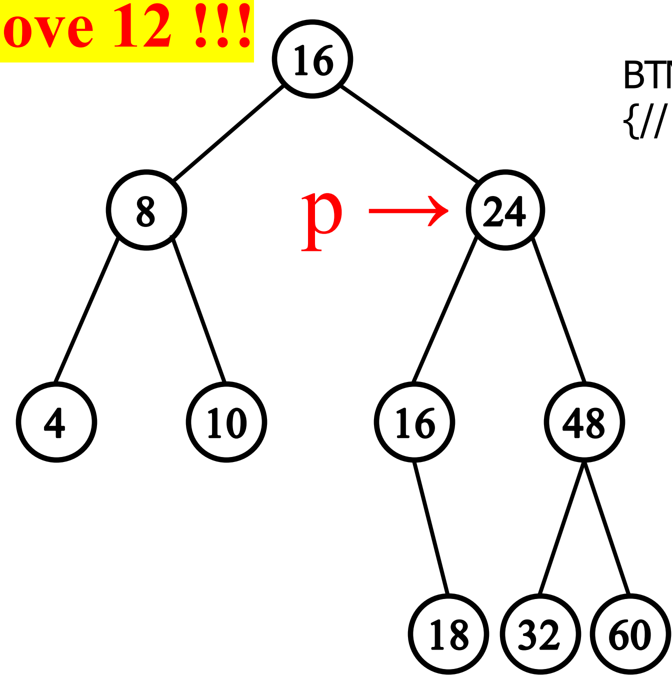
TmpPtr points to closest greater key than 12 (TmpPtr →16)

Red background: condition false

Green backgrnd: condition true

Removing a key at node with two children

Remove 12 !!!




```
BTNodeType *Remove (infotype *key, BTNodeType *p)
{// returns a pointer to node replacing the node removed.
      BTNodeType *TempPtr;
      if (p == NULL)
               return errorMessage(“empty tree”);
      if (key < p->data)
        p->left=Remove(key,p->left);
      else if (key > p->data)
        p->right=Remove(key,p->right);
      else                                    //key = p→data
        if (p->right && p->left) {       //double child node
          TmpPtr=findMin(p->right);
          p->data=TmpPtr->data;
          p->right=Remove(p->data,p->right); Call #1 point !!!
      }
      else {
          TmpPtr=p;
          if (p->left == NULL) p=p->right;
          else if (p->right == NULL) p=p->left;
          free(TmpPtr);
      }
      return p;
```

Call #1: recursive

p points to root of right subtree!

i.e., p → 24

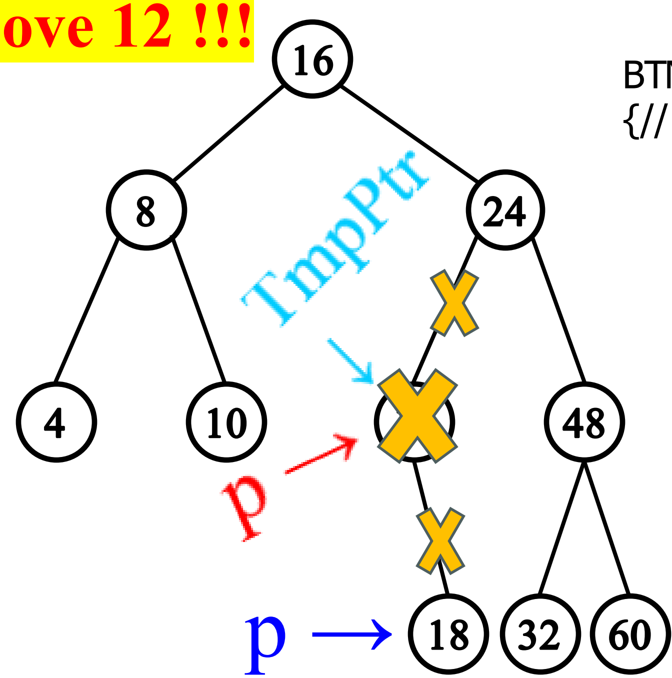
key(=16) < p→data (=24)

Red background: condition false

Green backgrnd: condition true

Removing a key at node with two children

Remove 12 !!!




```
BTNodeType *Remove (infotype *key, BTNodeType *p)
{// returns a pointer to node replacing the node removed.
    BTNodeType *TempPtr;
    if (p == NULL)
        return errorMessage("empty tree");
    if (key < p->data)
      p->left=Remove(key,p->left); Call #2 point !!!
    else if (key > p->data)
      p->right=Remove(key,p->right);
    else                                  //key = p→data
      if (p->right && p->left) {          //double child node
        TmpPtr=findMin(p->right);
        p->data=TmpPtr->data;
        p->right=Remove(p->data,p->right);
    }
    else {                                //single child node
        TmpPtr=p;
        if (p->left == NULL) p=p->right;
        else if (p->right == NULL) p=p->left;
        free(TmpPtr);
    }
    return p;
```

Call #2: recursive

p points to 16! i.e., p → 16

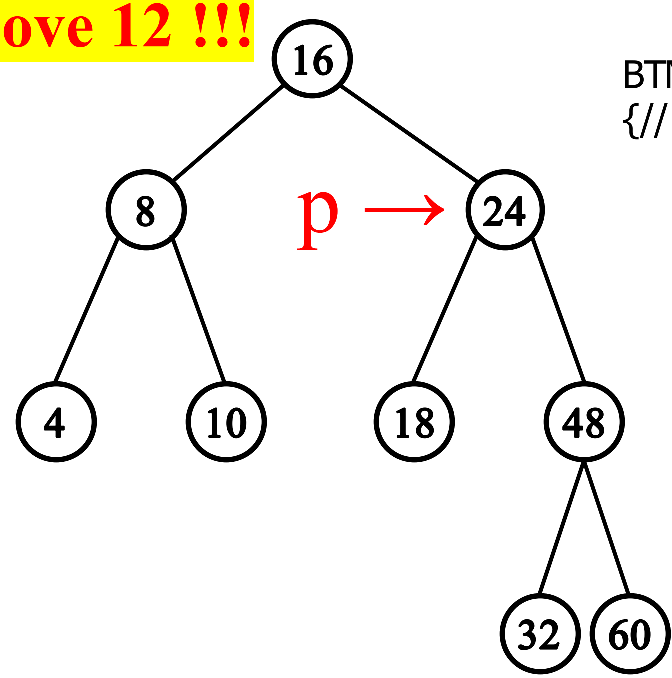
key(=16) = p→data (=16)

Red background: condition false

Green backgrnd: condition true

Removing a key at node with two children

Remove 12 !!!



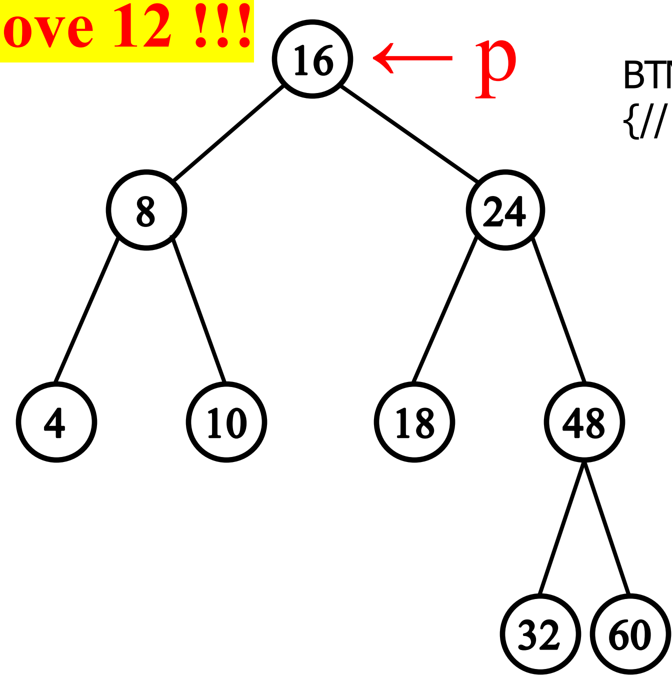

```
BTNodeType *Remove (infotype *key, BTNodeType *p)
{// returns a pointer to node replacing the node removed.
    BTNodeType *TempPtr;
    if (p == NULL)
            return errorMessage("empty tree");
    if (key < p->data)
      p->left=Remove(key,p->left); Call #2 point !!!
    else if (key > p->data)
      p->right=Remove(key,p->right);
    else                                //key = p→data
      if (p->right && p->left) {        //double child node
        TmpPtr=findMin(p->right);
        p->data=TmpPtr->data;
        p->right=Remove(p->data,p->right);
    }
    else {                              //single child node
        TmpPtr=p;
        if (p->left == NULL) p=p->right;
        else if (p->right == NULL) p=p->left;
        free(TmpPtr);
    }
    return p;
```

Return from Call #2 back to Call #1 at point of Call #2

Address of 18 returned to p→left with p pointing to 24.

Removing a key at node with two children

Remove 12 !!!



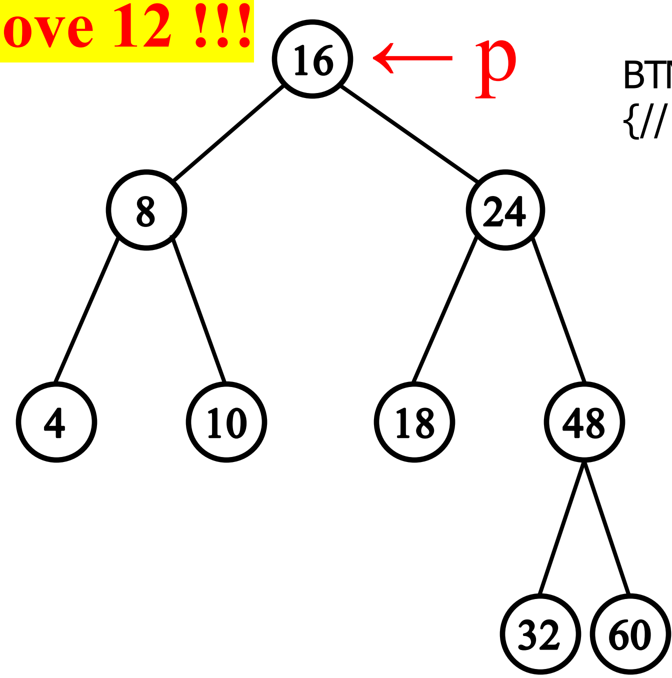

```
BTNodeType *Remove (infotype *key, BTNodeType *p)
{// returns a pointer to node replacing the node removed.
    BTNodeType *TempPtr;
    if (p == NULL)
          return errorMessage("empty tree");
    if (key < p->data)
      p->left=Remove(key,p->left);
    else if (key > p->data)
      p->right=Remove(key,p->right);
    else                                  //key = p→data
      if (p->right && p->left) {          //double child node
         TmpPtr=findMin(p->right);
         p->data=TmpPtr->data;
         p->right=Remove(p->data,p->right); Call #1 point !!!
    }
    else {                                //single child node
         TmpPtr=p;
         if (p->left == NULL) p=p->right;
         else if (p->right == NULL) p=p->left;
         free(TmpPtr);
    }
    return p;
```

Return from Call #1 back to Call #0 at the point of Call #1

Address of 24 returned to p→right with p pointing to 16 (12 gone!!!).

Removing a key at node with two children

Remove 12 !!!



Return from Call #0

Address of 16 (i.e., modified BST) returned to the invoker function with 12 removed.

```
BTNodeType *Remove (infotype *key, BTNodeType *p)
{// returns a pointer to node replacing the node removed.
      BTNodeType *TempPtr;
      if (p == NULL)
               return errorMessage("empty tree");
      if (key < p->data)
        p->left=Remove(key,p->left);
      else if (key > p->data)
        p->right=Remove(key,p->right);
      else                                  //key = p→data
        if (p->right && p->left) {          //double child node
           TmpPtr=findMin(p->right);
           p->data=TmpPtr->data;
           p->right=Remove(p->data,p->right);
      }
      else {                                //single child node
           TmpPtr=p;
           if (p->left == NULL) p=p->right;
           else if (p->right == NULL) p=p->left;
           free(TmpPtr);
      }
      return p;
```

Supplementary slides…

Calculation of Access Time

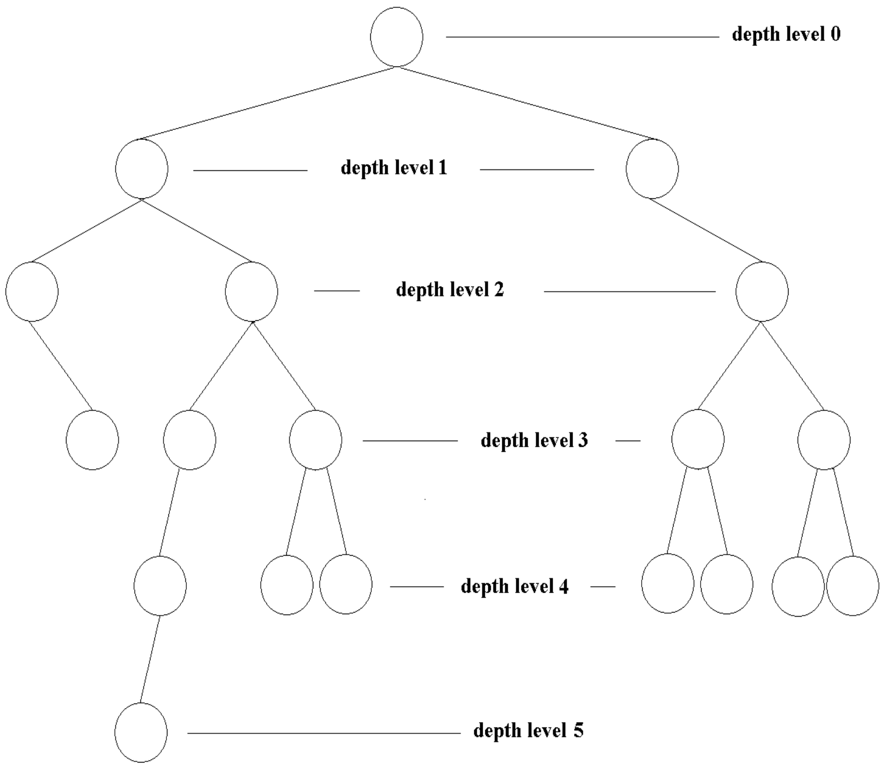
- Access to some specific piece of data is achieved when the node this piece of data resides is found. To find this node, a search gets started from the root.
- Data at each node is compared with the key that is searched for.
- Counting up the number of comparisons for each node will render the access time.

Best/Average/Worst (B/A/W) Case Access Times of a specific BST

- The best or fastest access time **of a specific BST** describes the access time to a node that can be attained with the smallest number of comparisons in this BST. Obviously this node is the root of this BST.
- The average access time **of a specific BST** is total number of comparisons required to find all nodes divided by the number of nodes in this BST.
- The worst case access time **of a specific BST** is the number of nodes on the longest path from the root to a leaf in this BST. It is equal to the height of this tree +1.
- **Important:** !!! Please note that B/A/W topologies and B/A/W case access times are two different concepts!!!

Calculation of Access Times

Depth level	# of Comparisons/ node	# of Comparisons/ level
0	1	1*1=1
1	2	2*2=4
2	3	3*3=9
3	4	4*5=20
4	5	5*7=35
5	6	6*1=6
Total	Nodes: 19	75



BST Access Times	
Average	75/19=3.94
Worst	6
Best	1

Best/Average/Worst BST Topologies

- In the analysis of Find function we considered *three different topology types*:
  - ***Average case***: this type includes all possible BST topologies from best to worst case topologies;
  - ***Best case***: this type includes *all BST topologies** that provide the shortest (best) total or average access time;
  - ***Worst case***: this type includes *all BST topologies** that provide the longest (worst) total or average access time which is simply a LL.

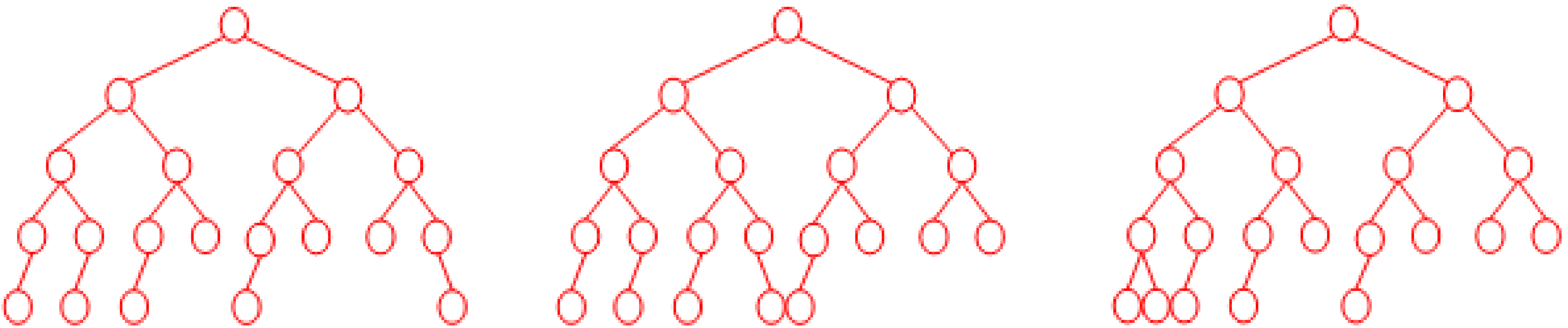
* ***Is there more than one best/worst topology? Look at next slide!***

Best/Worst BST Topologies

- A best (worst) BST topology is one with the best total or average access time.
- Each topology with all depth levels filled up with the only possible exception of the last one is a best case topology.
- Each different arrangement combination of leaves at the last depth level refers to a different BST topology that provides the same shortest total search time; hence a best case BST topology.
- Examples are at the next slide.

20-Node Best BST Topologies

We see three examples to such BST topologies…



Question: How many such BST topologies are there?
Solution in the following slides…

Solution

- Suppose that at depth level $d-1$ (DL $d-1$) and DL d we have $\frac{k}{2}$ and i nodes, respectively. Then at DL d we have k possible positions (i.e., k is the total number of child pointers of $\frac{k}{2}$ nodes in DL $d-1$ to place these i nodes. Then we may place these i nodes in k possible positions in $\binom{k}{i}$ different ways, each yielding a different BST topology. Calculating this for all DLs and multiply the combinations for each DL will yield the total number of different BST topologies or

$$\prod_{d=0}^{n-1} \binom{2 * number\ of\ nodes\ in\ d}{the\ number\ of\ nodes\ in\ d+1}$$

Solution

Now let's answer the question:

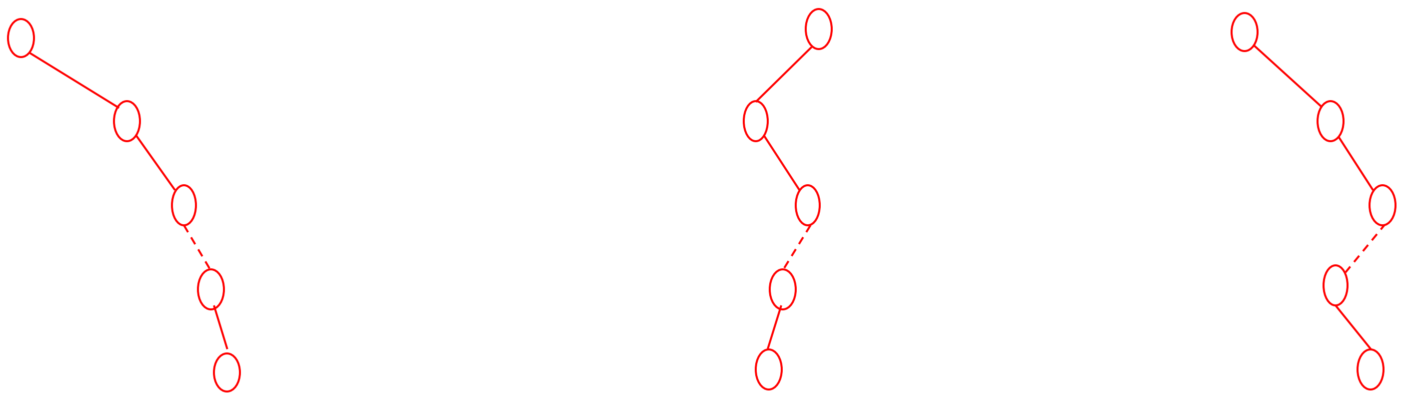
- In DL0 (the first DL), we have one possible root position for the root: $\binom{1}{1}$
- DL1: two possible positions (both children of root) for its two nodes: $\binom{2}{2}$
- DL2: Four possible positions for its four nodes: $\binom{4}{4}$
- DL3: Eight possible positions for eight nodes: $\binom{8}{8}$
- DL4: In this last DL we have 20-($1+2+4+8=\sum_{d=0}^{3} 2^i = 2^4 - 1 = 15$)=5 nodes and $2*8=16$ positions to place these 5 nodes: $\binom{16}{5}$ or

$$\prod_{d=0}^{n-1}\binom{2*number\ of\ nodes\ in\ d}{the\ number\ of\ nodes\ in\ d+1} = \binom{1}{1}\binom{2}{2}\binom{4}{4}\binom{8}{8}\binom{16}{5} = \binom{16}{5}$$

$\Longrightarrow$ there are $\binom{16}{5}$ 20-node best BST topologies.

n-Node Worst BST Topologies

We see three examples to such BST topologies…



Exercise: Find how many such BST topologies there are. Use the same technique as that to find the number of the best BST topologies…