Data Structures – Week #6

Special Trees

Outline

- Adelson-Velskii-Landis (AVL) Trees
- Splay Trees
- B-Trees

AVL Trees

Motivation for AVL Trees

- Accessing a node in a BST takes *$O(\log_2 n)$* in average.
- A BST can be structured so as to have an average access time of *$O(n)$*. *Can you think of one such BST?*
- Q: Is there a way to *guarantee a worst-case access time of $O(\log_2 n)$ per node* or can we find a way to *guarantee a BST depth of $O(\log_2 n)$*?
- A: ***AVL Trees***

Definition

An *AVL tree* is a *BST* with the following *balance condition*:

for each node in the BST, the height of left and right sub-trees can differ by at most 1, or

$$\left| h_{N_L} - h_{N_R} \right| \leq 1.$$

Remarks on Balance Condition

- *Balance condition must be easy to maintain*:
  - This is the reason, for example, for the balance condition's not being as follows: the height of left and right sub-trees of each node have the same height.
- *It ensures the depth of the BST is $O(\log_2 n)$.*
- The *height information is stored* as an additional field in BTNodeType.

Structure of an AVL Tree

```
struct BTNodeType {
    infoType *data;
    unsigned int height;
    struct BTNodeType *left;
    struct BTNodeType *right;
}
```

Rotations

Definition:

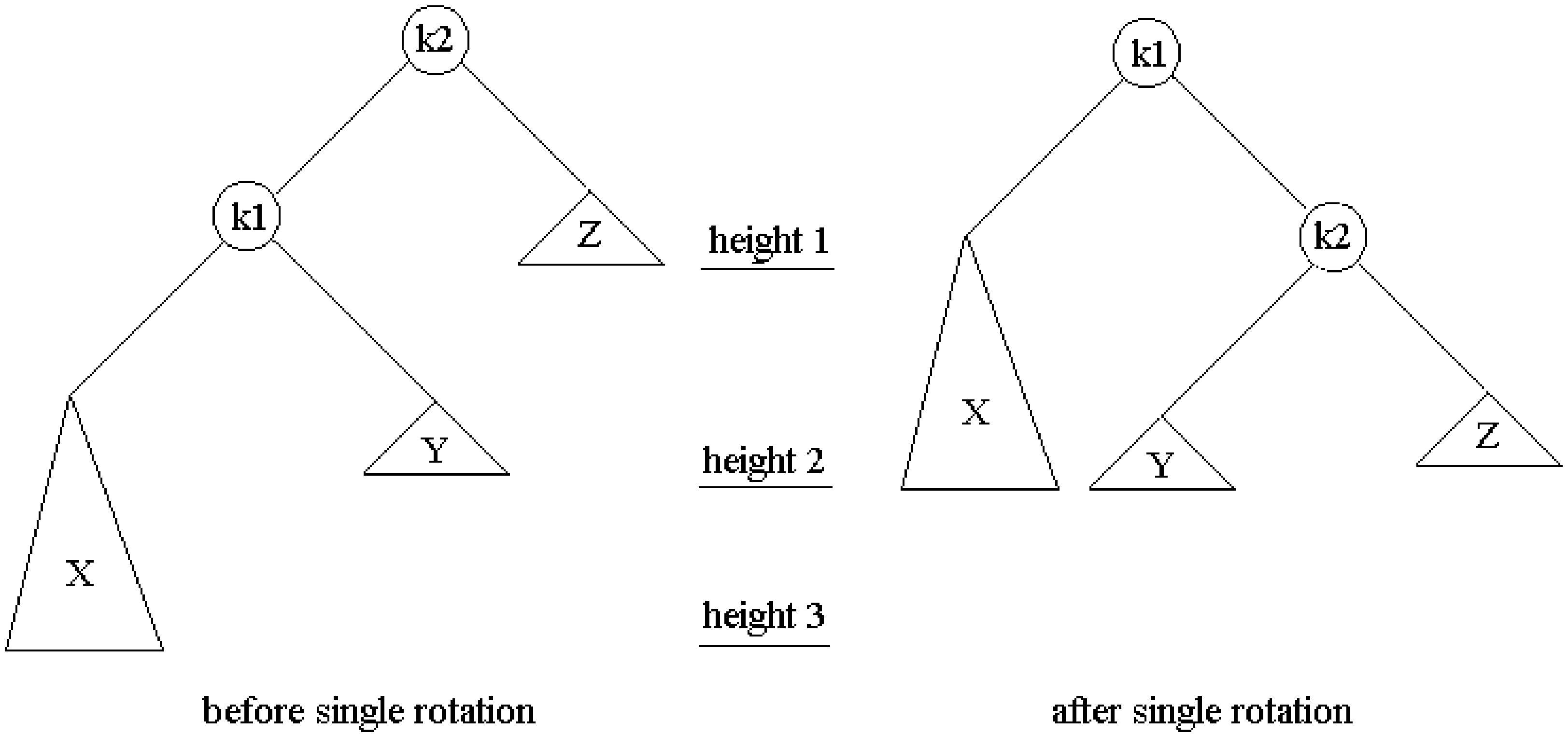
- *Rotation* is the operation performed on a BST to restore its AVL property lost as a result of an insert operation.

- We consider the node α whose new balance violates the AVL condition.

Rotation

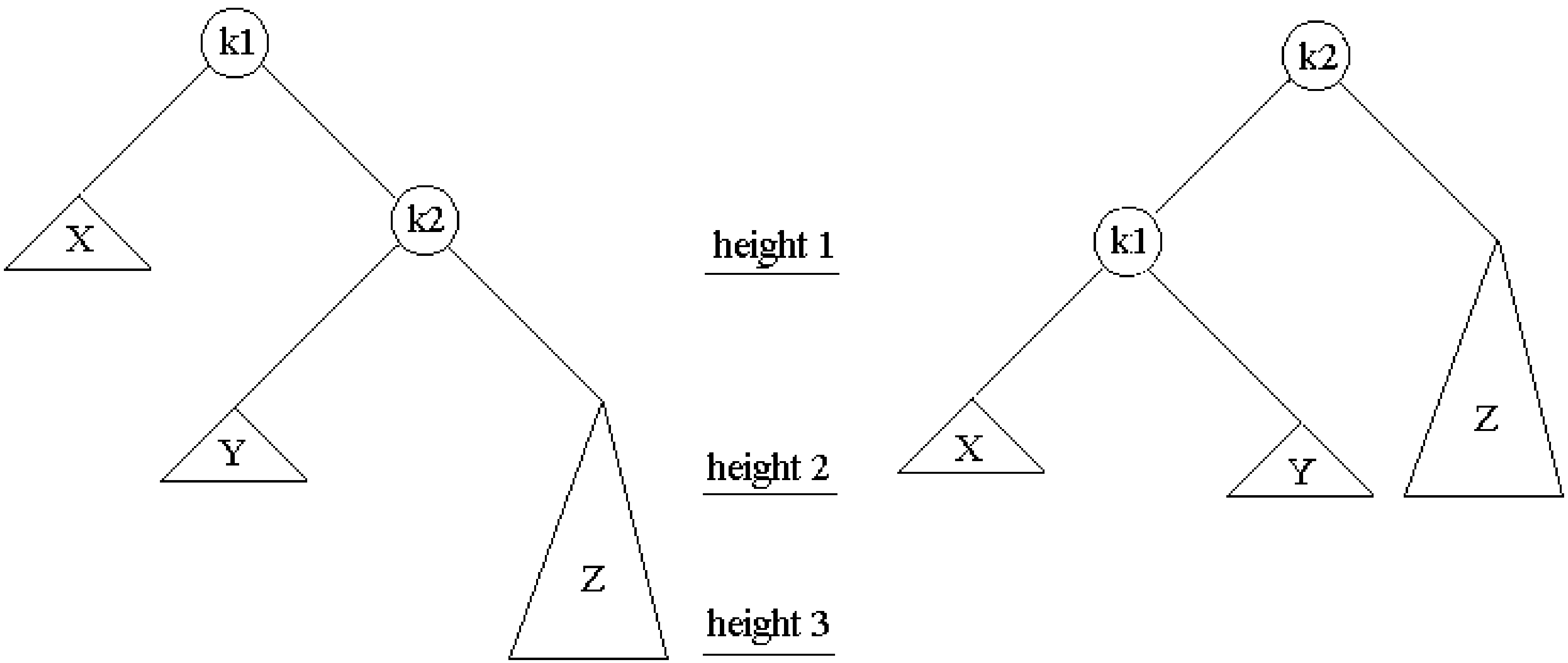
- Violation of AVL condition
- The AVL condition violation may occur in four cases:
  - Insertion into *left subtree of the left child (L/L)*
  - Insertion into *right subtree of the left child (R/L)*
  - Insertion into *left subtree of the right child (L/R)*
  - Insertion into *right subtree of the right child (R/R)*
- The outside cases 1 and 4 (i.e., L/L and R/R) are fixed by a *single rotation*.
- The other cases (i.e., R/L and L/R) need two rotations called *double rotation* to get fixed.
- These are fundamental operations in balanced-tree algorithms.

Single Rotation (L/L)



$\alpha \equiv$ k2 node

Single Rotation (R/R)



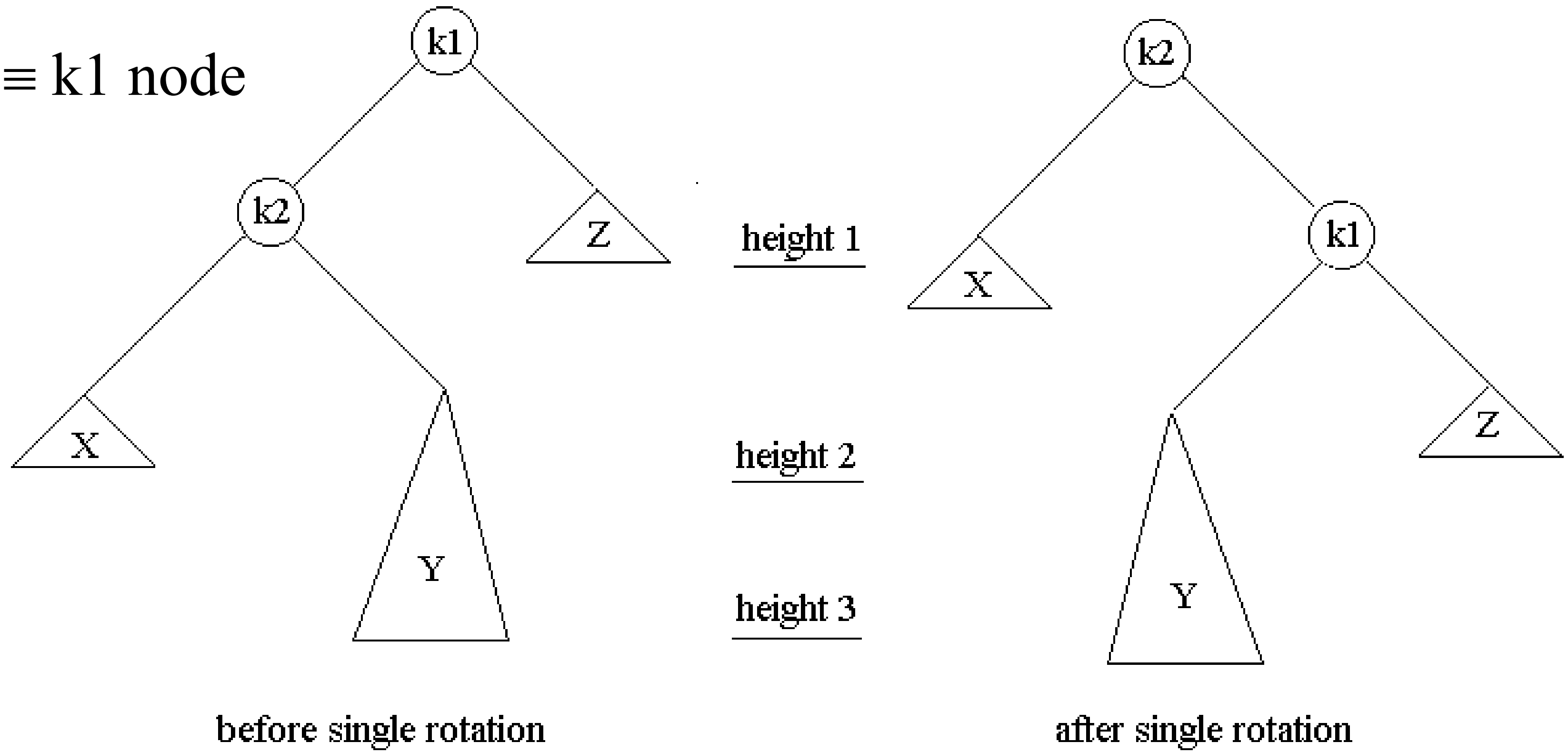
before single rotation

after single rotation

$\alpha \equiv$ k1 node

Double Rotation (R/L)

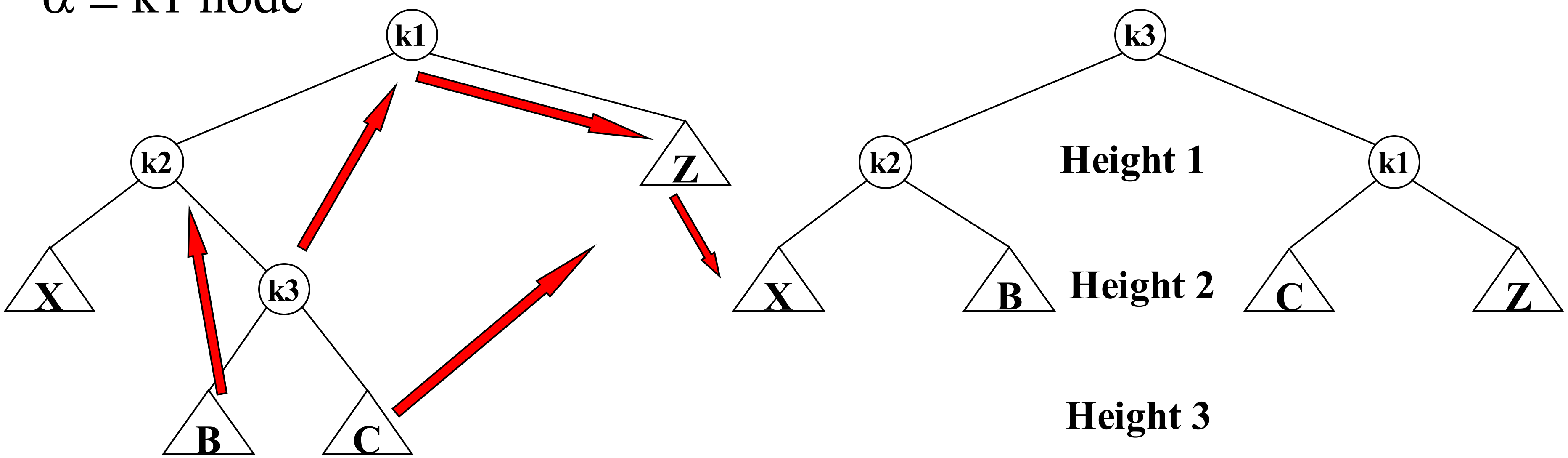
$\alpha \equiv$ k1 node



Single rotation cannot fix the AVL condition violation!!!

Double Rotation (R/L)

$\alpha \equiv$ k1 node



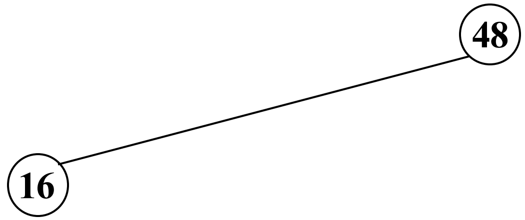
The symmetric case (L/R) is handled similarly left as an exercise to you!

Constructing an AVL Tree – Animation

48

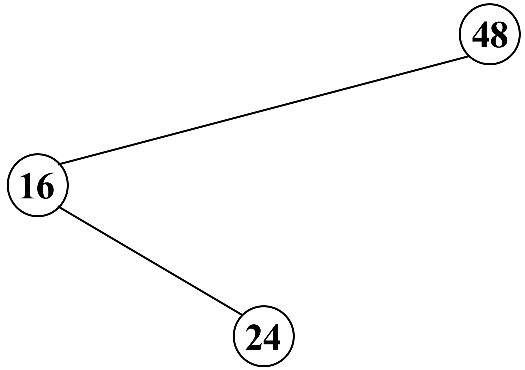
Constructing an AVL Tree – Animation

48 16



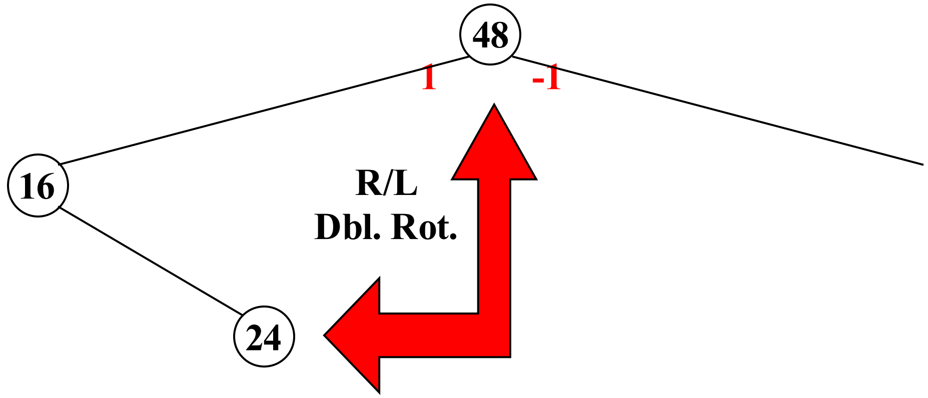
Constructing an AVL Tree – Animation

48 16 24



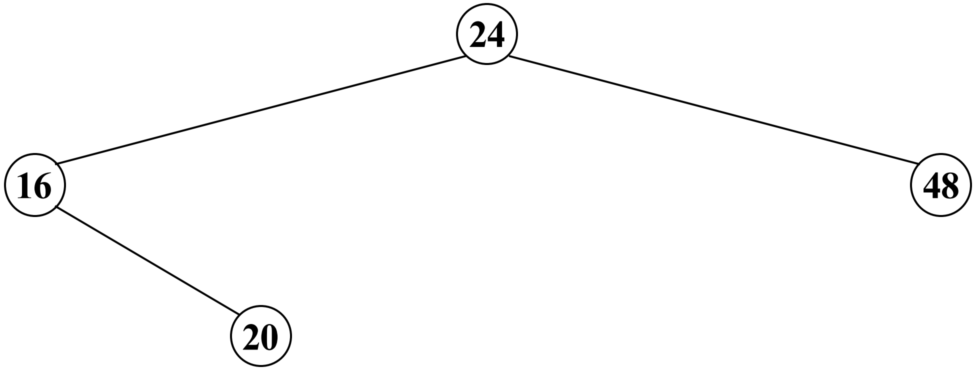
Constructing an AVL Tree – Animation

48 16 24



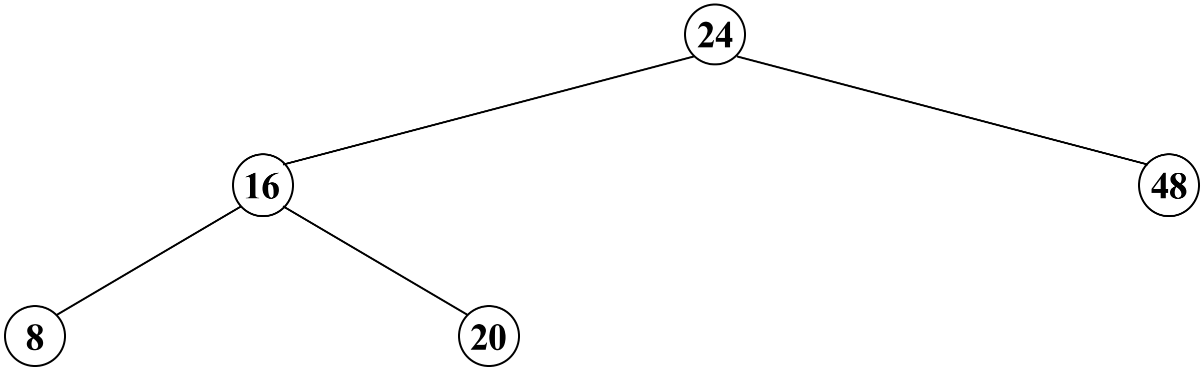
Constructing an AVL Tree – Animation

48 16 24 20



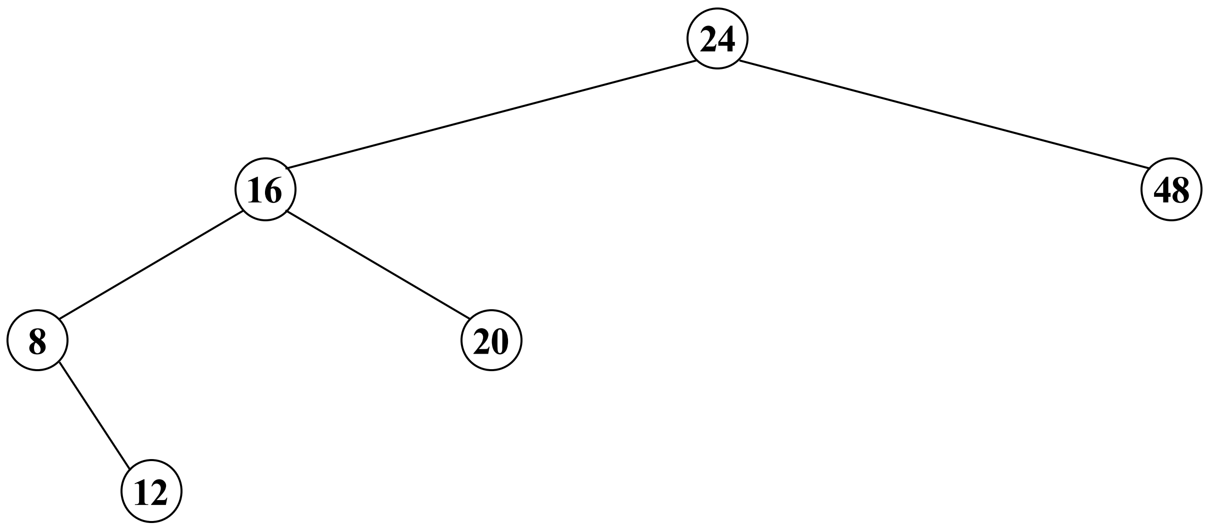
Constructing an AVL Tree – Animation

48 16 24 20 8



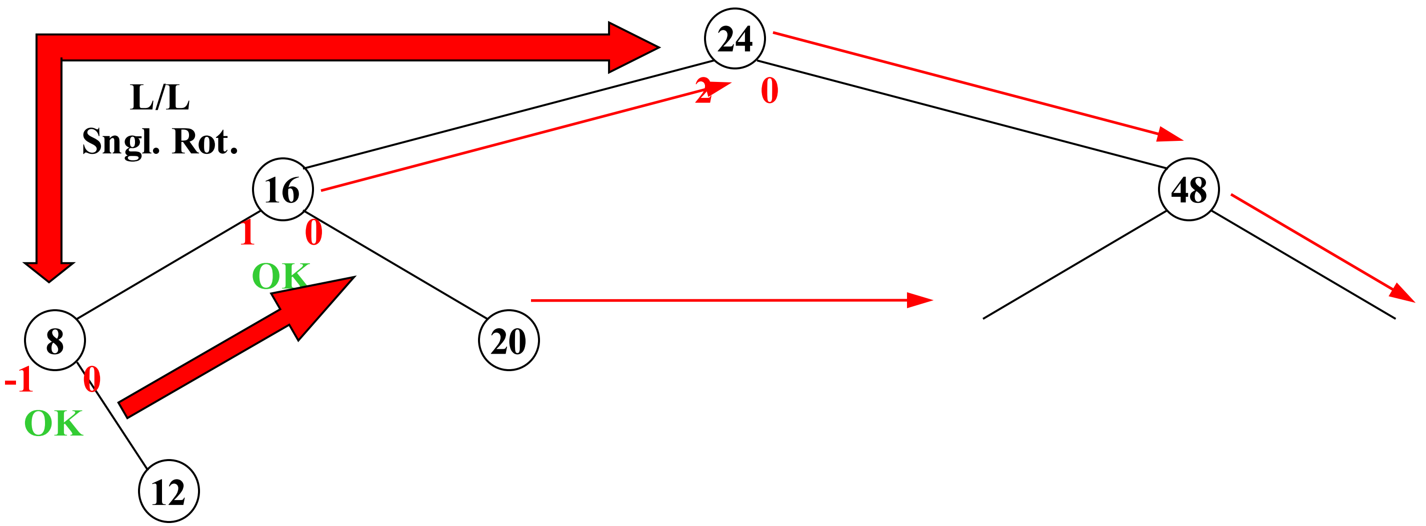
Constructing an AVL Tree – Animation

48 16 24 20 8 12



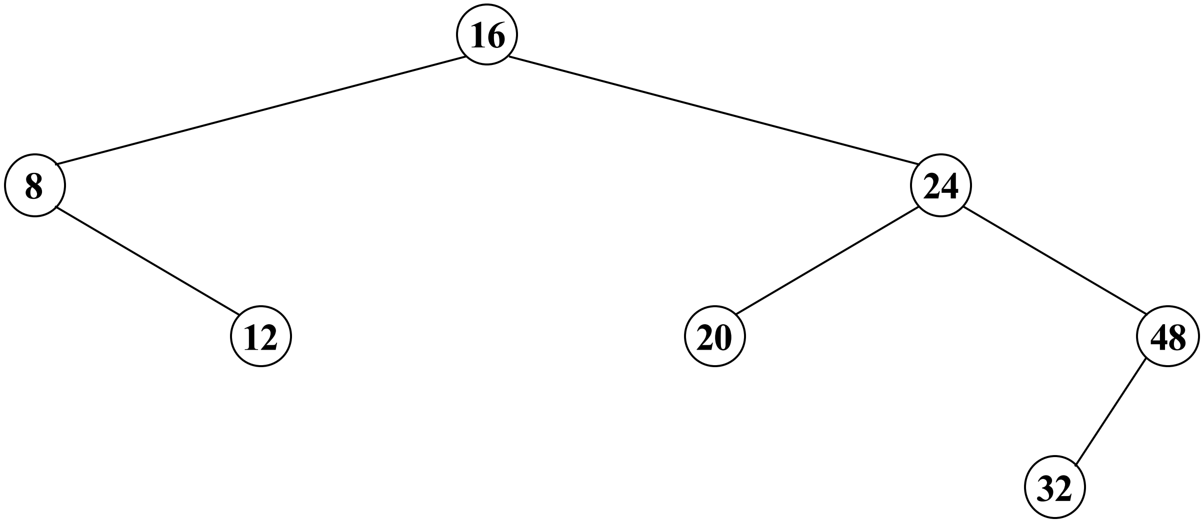
Constructing an AVL Tree – Animation

48	**16**	**24**	**20**	**8**	**12**



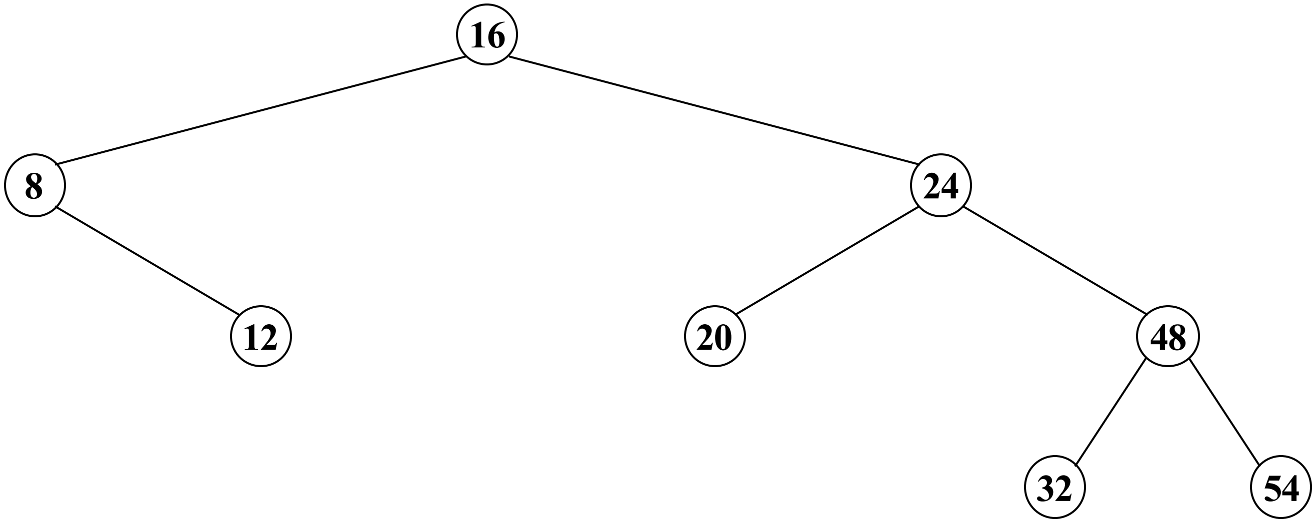
Constructing an AVL Tree – Animation

48 16 24 20 8 12 32



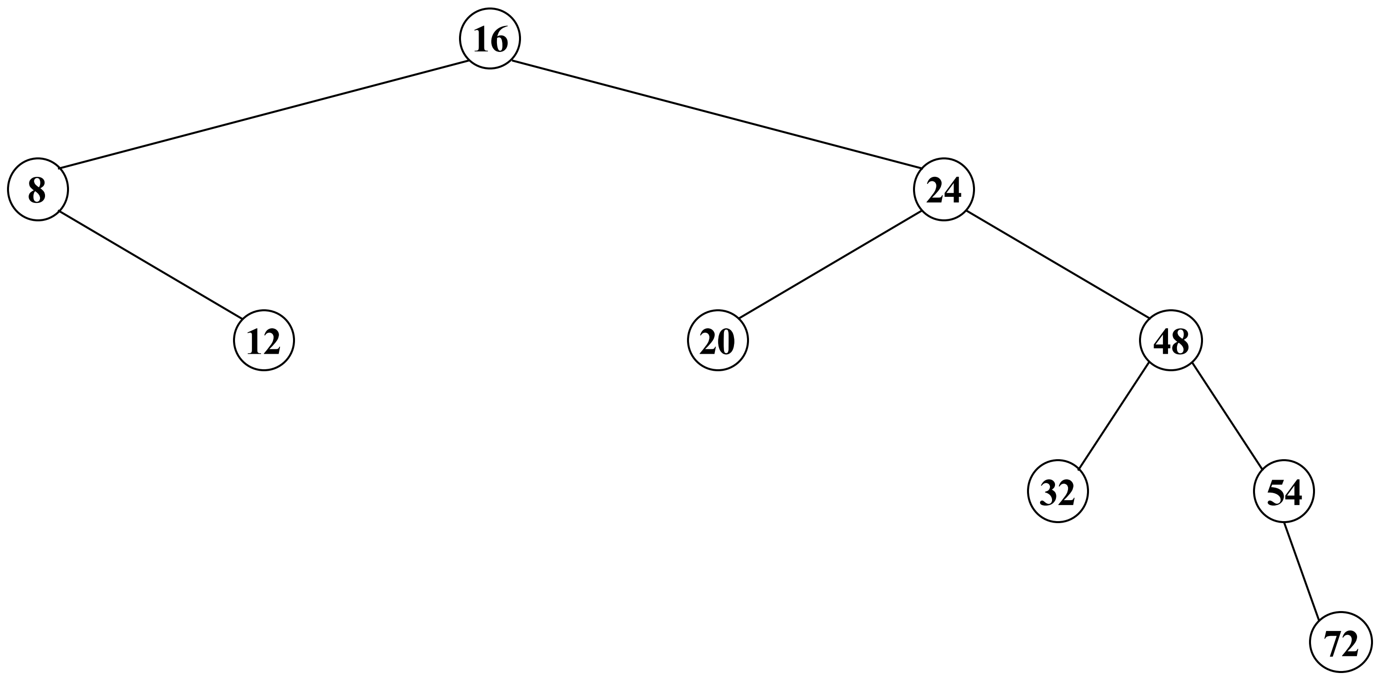
Constructing an AVL Tree – Animation

48 16 24 20 8 12 32 54



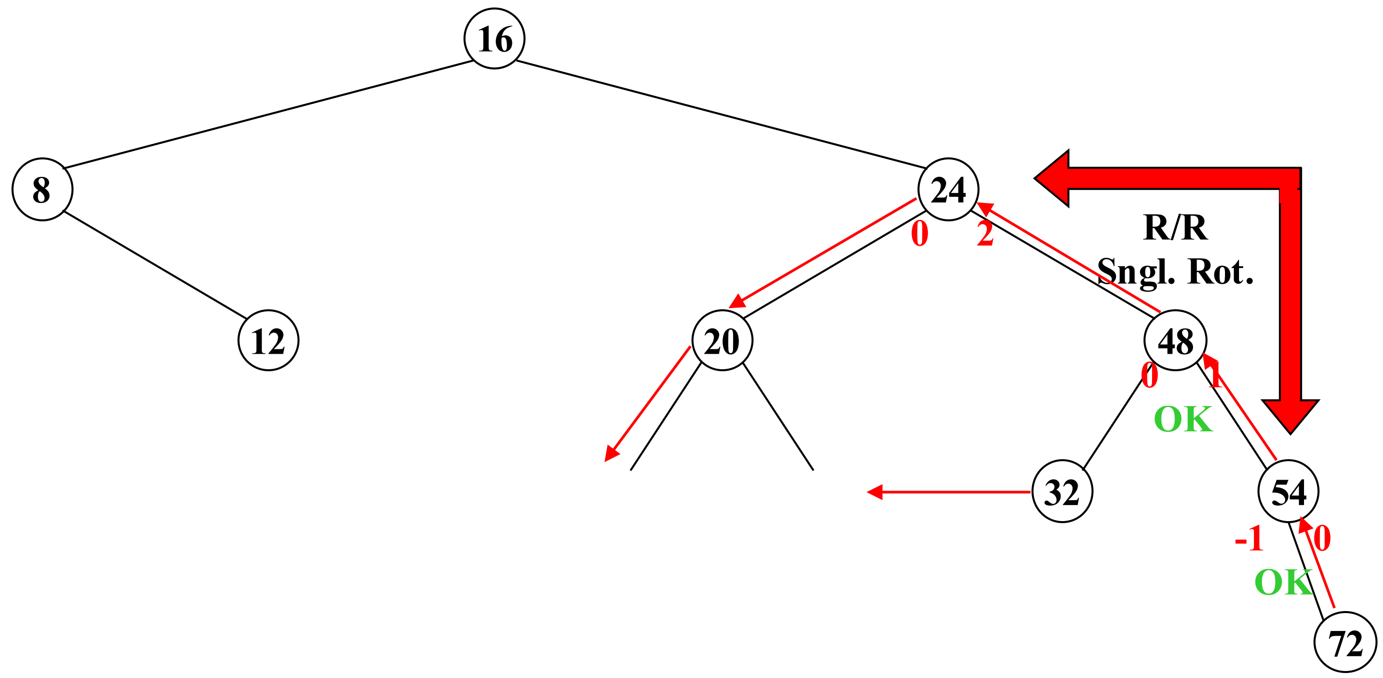
Constructing an AVL Tree – Animation

48 16 24 20 8 12 32 54 72



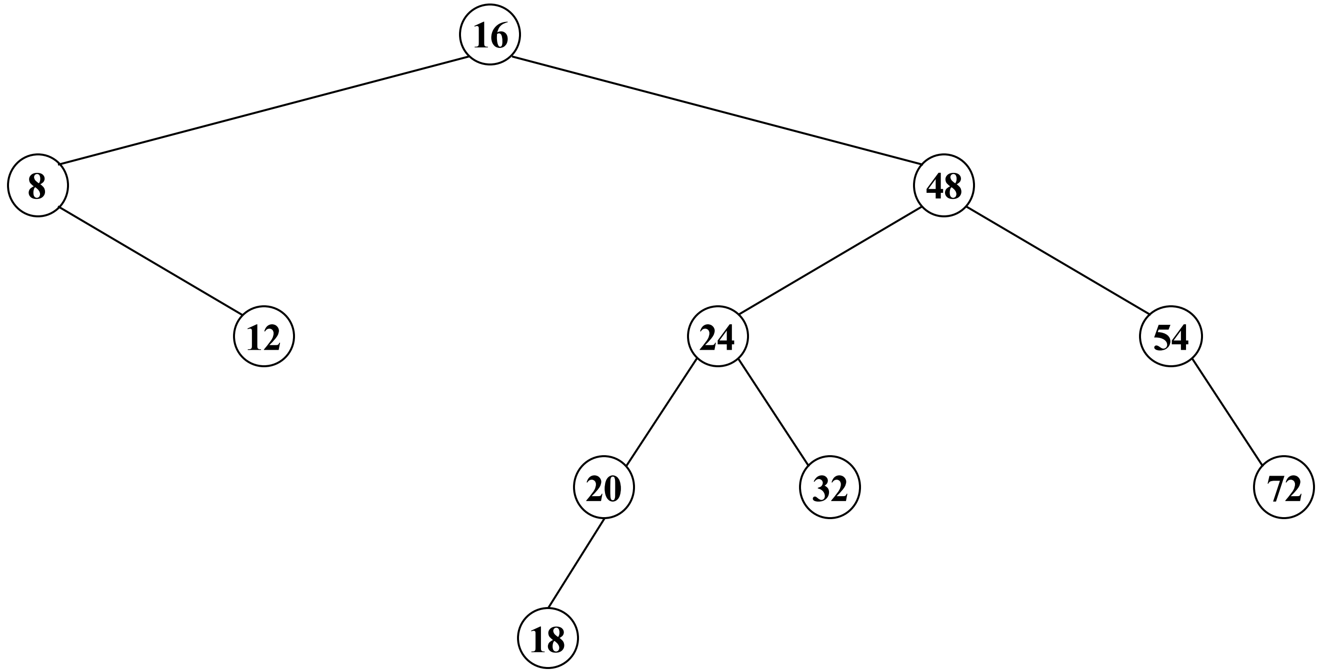
Constructing an AVL Tree – Animation

48	**16**	**24**	**20**	**8**	**12**	**32**	**54**	**72**



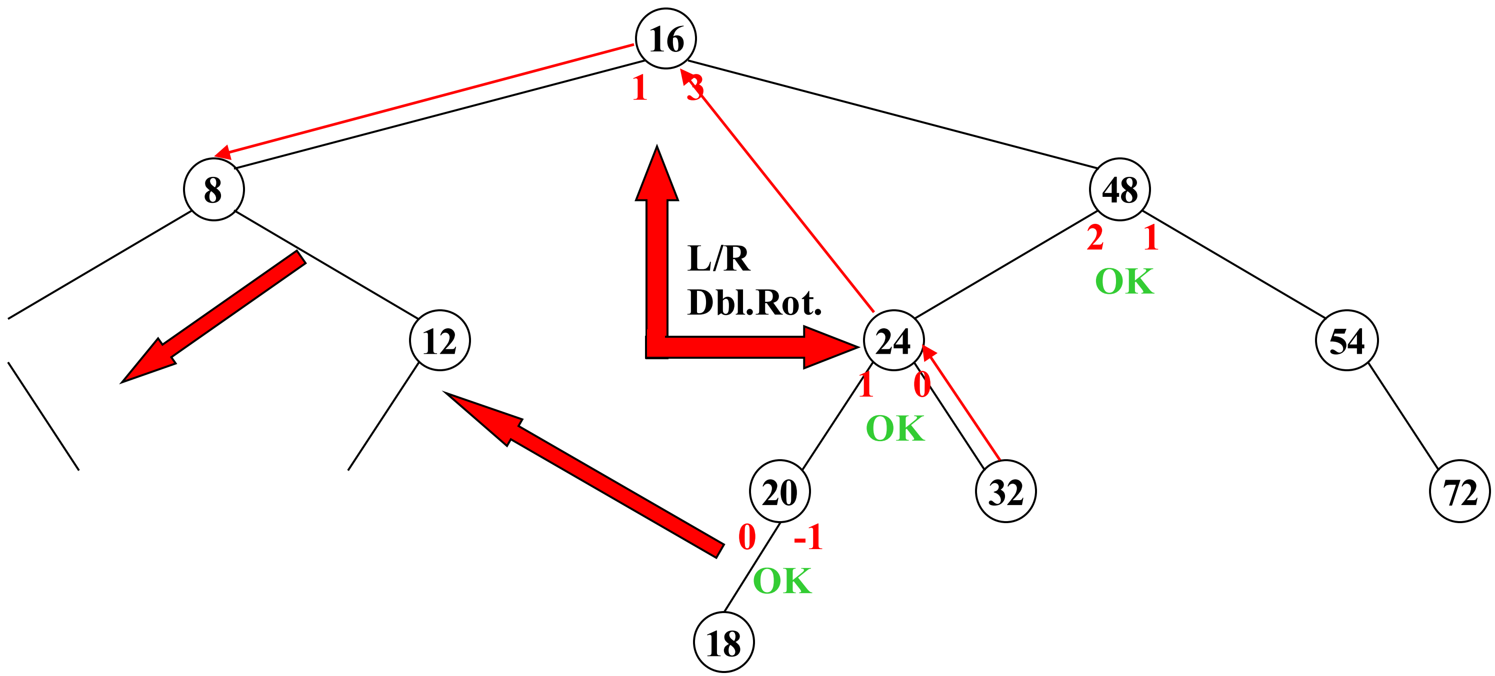
Constructing an AVL Tree – Animation

48 16 24 20 8 12 32 54 72 18



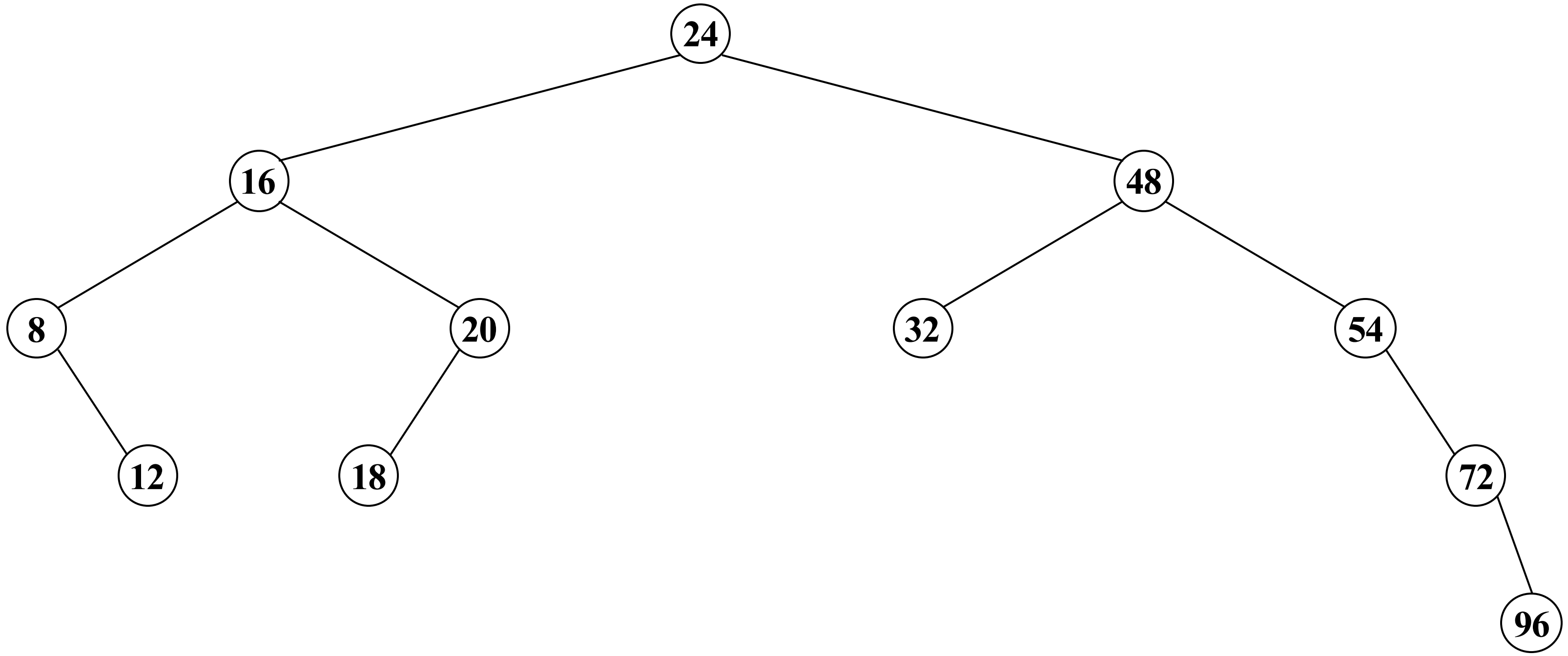
Constructing an AVL Tree – Animation

48 16 24 20 8 12 32 54 72 18



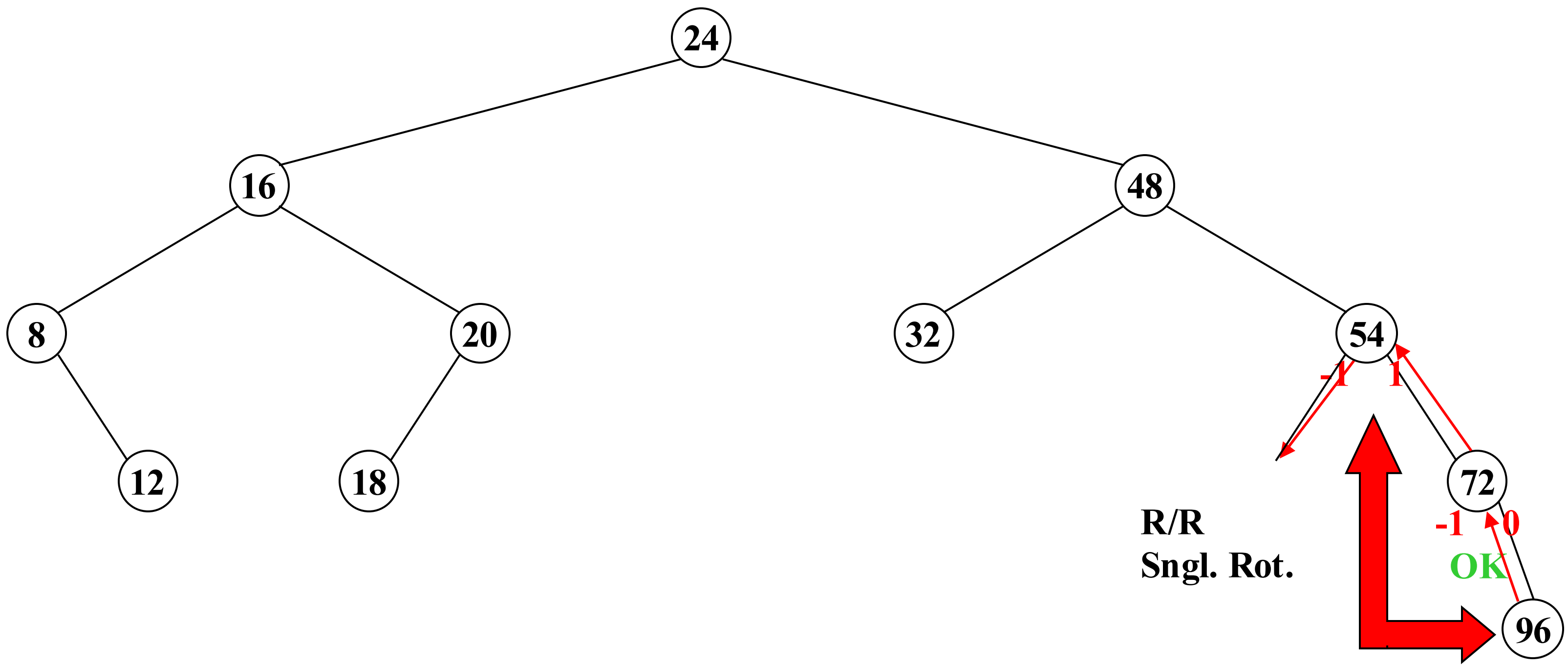
Constructing an AVL Tree – Animation

48 16 24 20 8 12 32 54 72 18 96

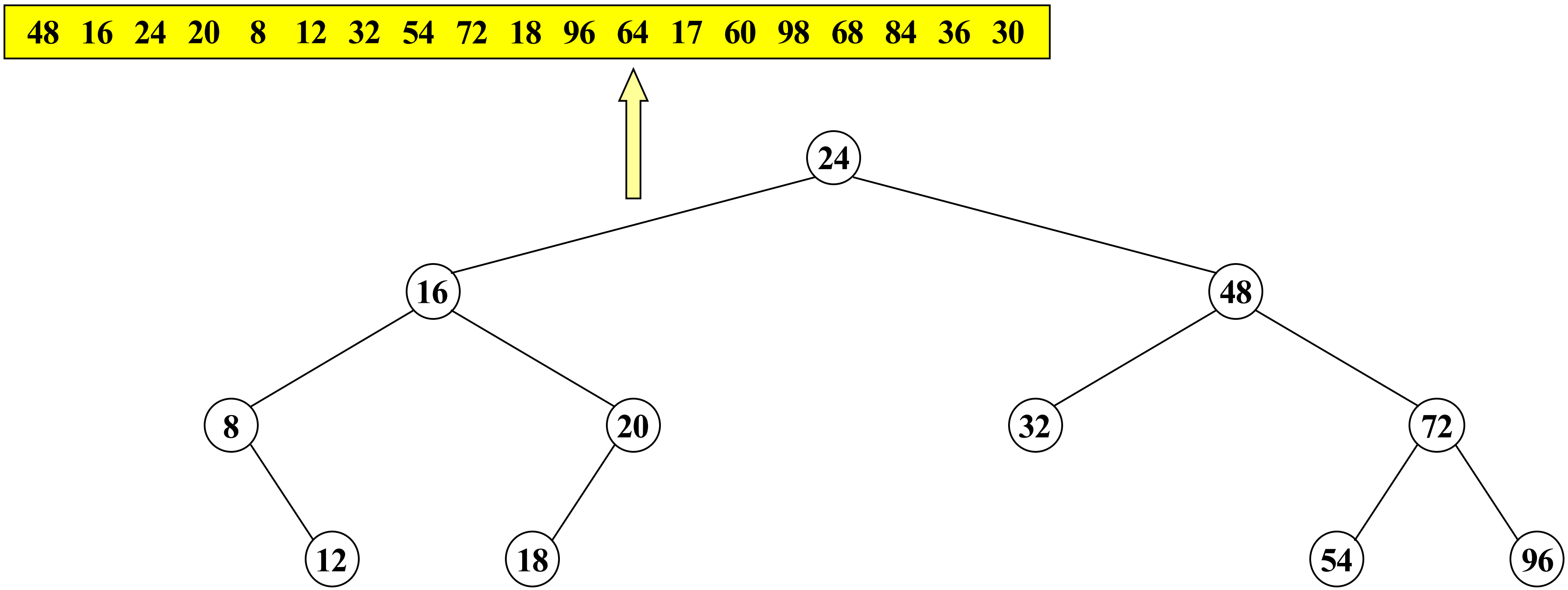


Constructing an AVL Tree – Animation

48 16 24 20 8 12 32 54 72 18 96



Constructing an AVL Tree – Animation



Height versus Number of Nodes

- The *minimum number* of nodes in an AVL tree recursively relates to the height of the tree as follows:

$$S(h) = S(h-1) + S(h-2) + 1;$$

Initial Values: $S(0)=1;\ S(1)=2$

Homework: Solve for $S(h)$ as a function of h!

Splay Trees

Motivation for Splay Trees

- We are looking for a data structure where, *even though some worst case (O(n)) accesses may be possible*, *m consecutive tree operations starting from an empty tree (inserts, finds and/or removals) take* $O(m*\log_2 n)$.
- Here, the main idea is to assume that, *O(n) accesses* are not bad as long as they *occur relatively infrequently*.
- Hence, we are looking for *modifications of a BST per tree operation that attempts to minimize O(n) accesses*.

Splaying

- The underlying idea of splaying is to *move a deep node accessed upwards to the root*, assuming that it will be accessed in the near future again.
- While doing this, other deep nodes are also carried up to smaller depth levels, making the average depth of nodes closer to $O(\log_2 n)$.

Splaying

- Splaying is similar to bottom-up AVL rotations
- If a node *X is the child of the root* R,
  - then we *rotate only X and R, and this is the last rotation performed*.

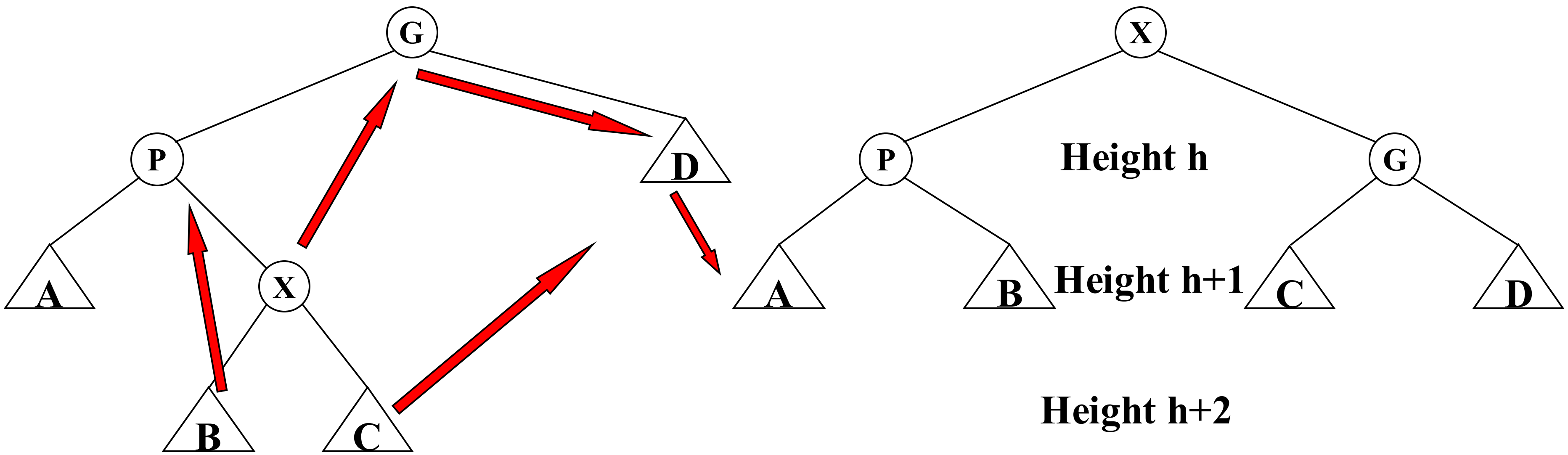
    else consider *X*, its *parent P* and *grandparent G*.

    Two cases and their symmetries to consider

    Zig-zag case, and

    Zig-zig case.

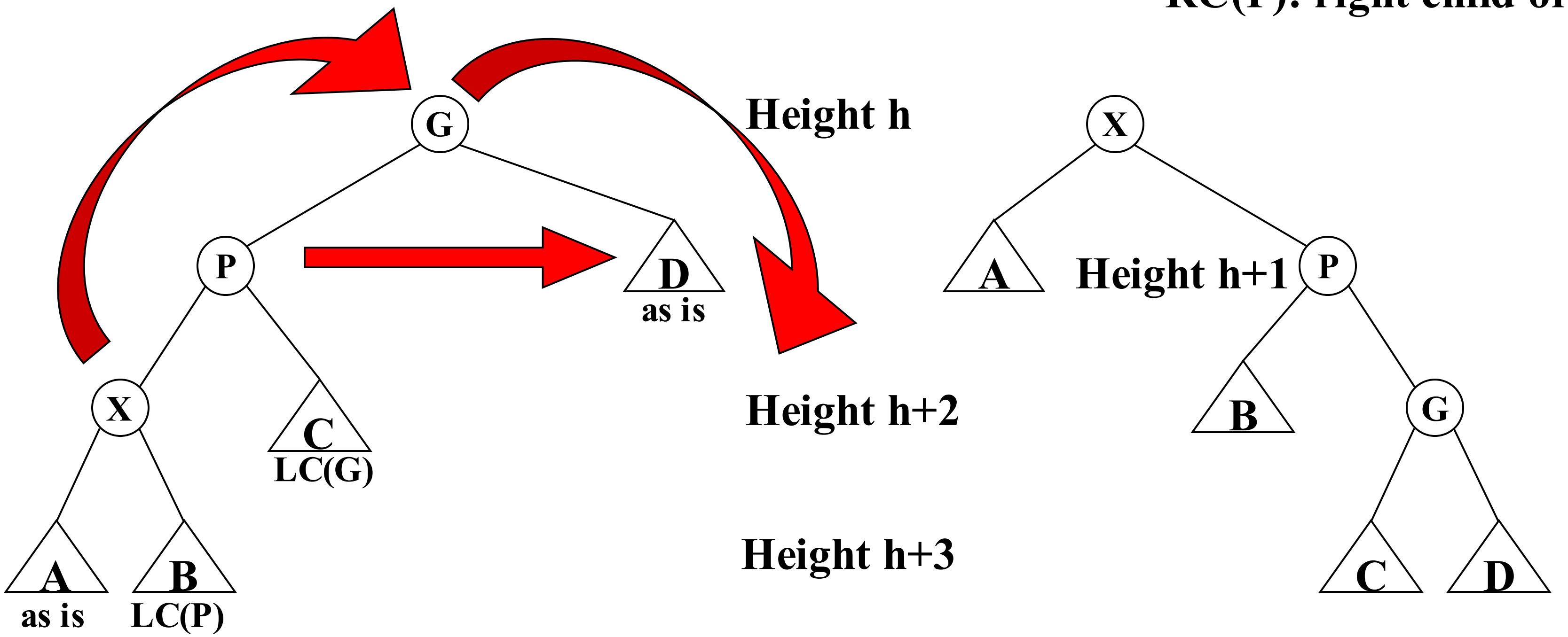
Zig-zag case



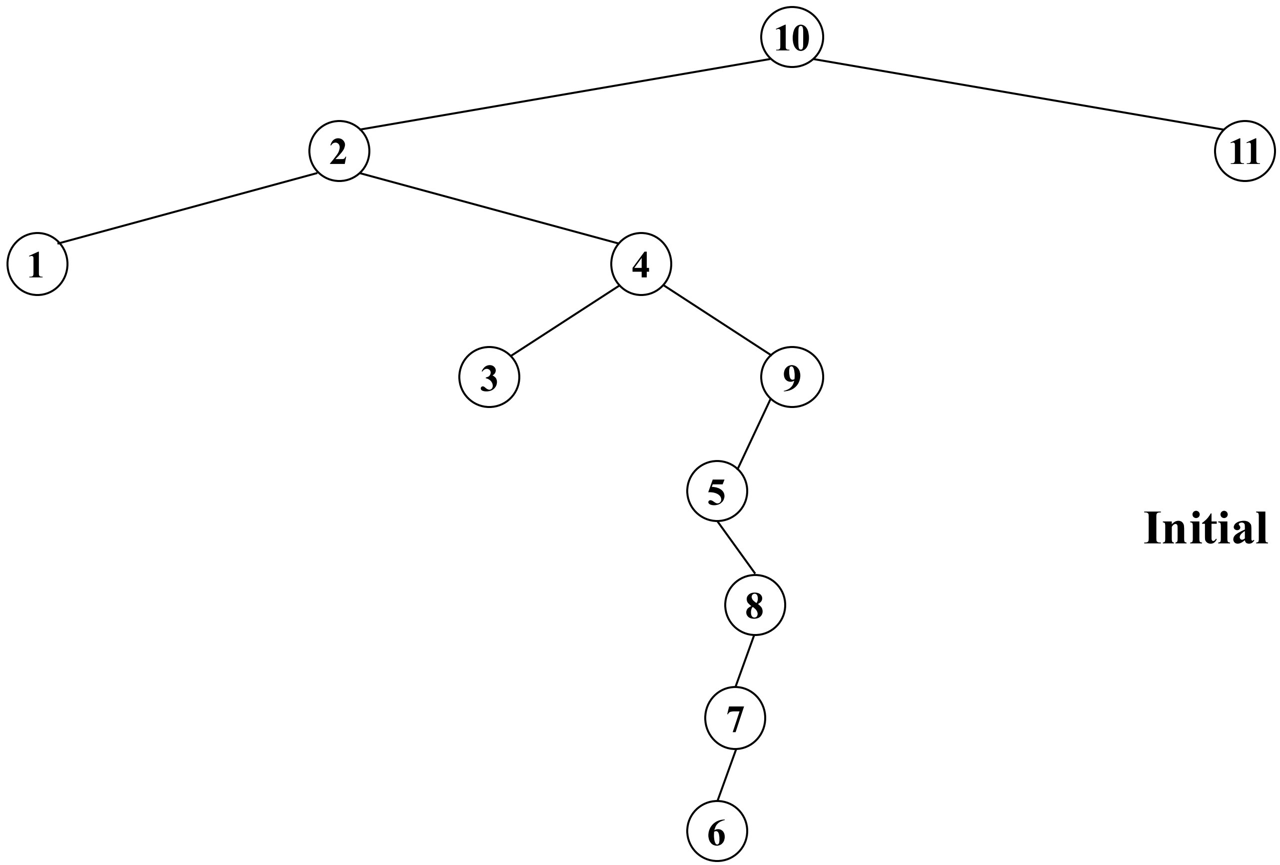
This is the same operation as an AVL double rotation in an R/L violation.

Zig-zig case

LC(P): left child of node P
RC(P): right child of node P

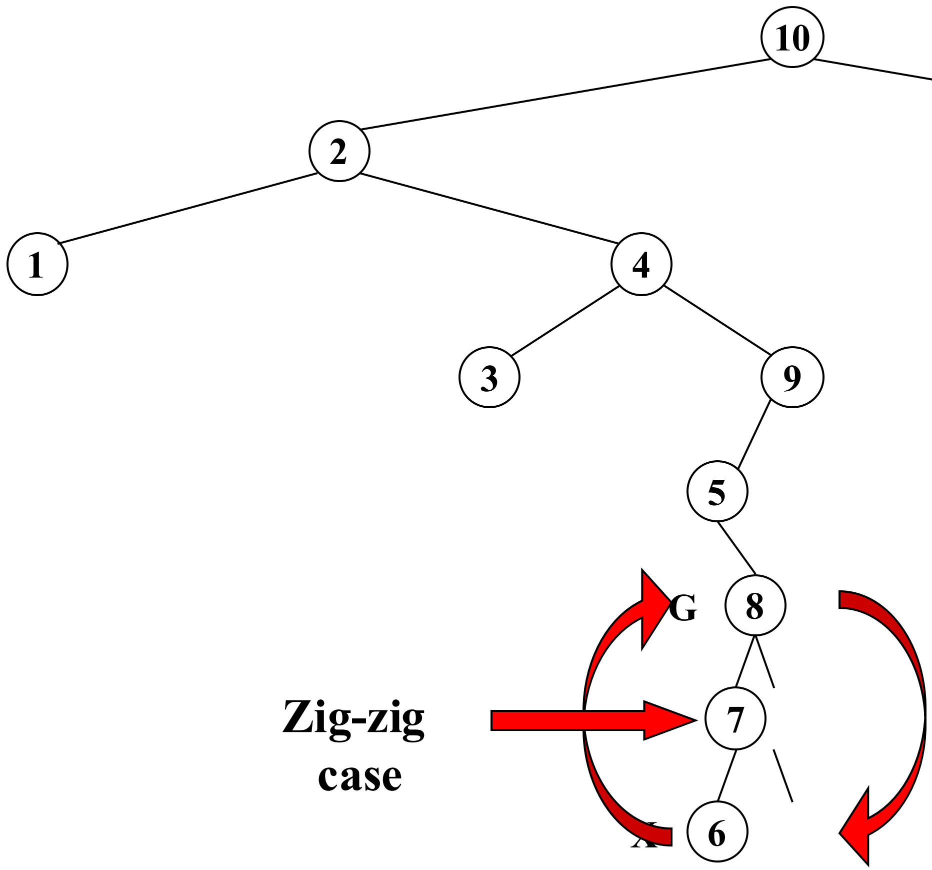


Animated Example



Initial BST

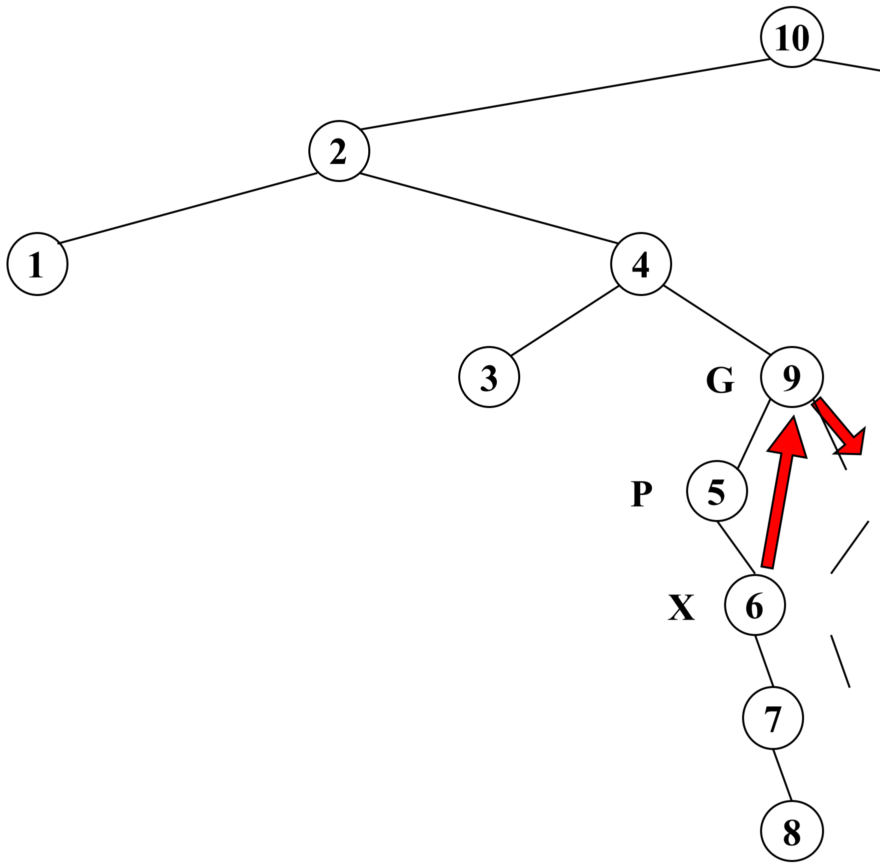
Animated Example



Node with 6 accessed!

G stands for **<u>G</u>randparent**
P stands for **<u>P</u>arent**
X denotes the ***current node***

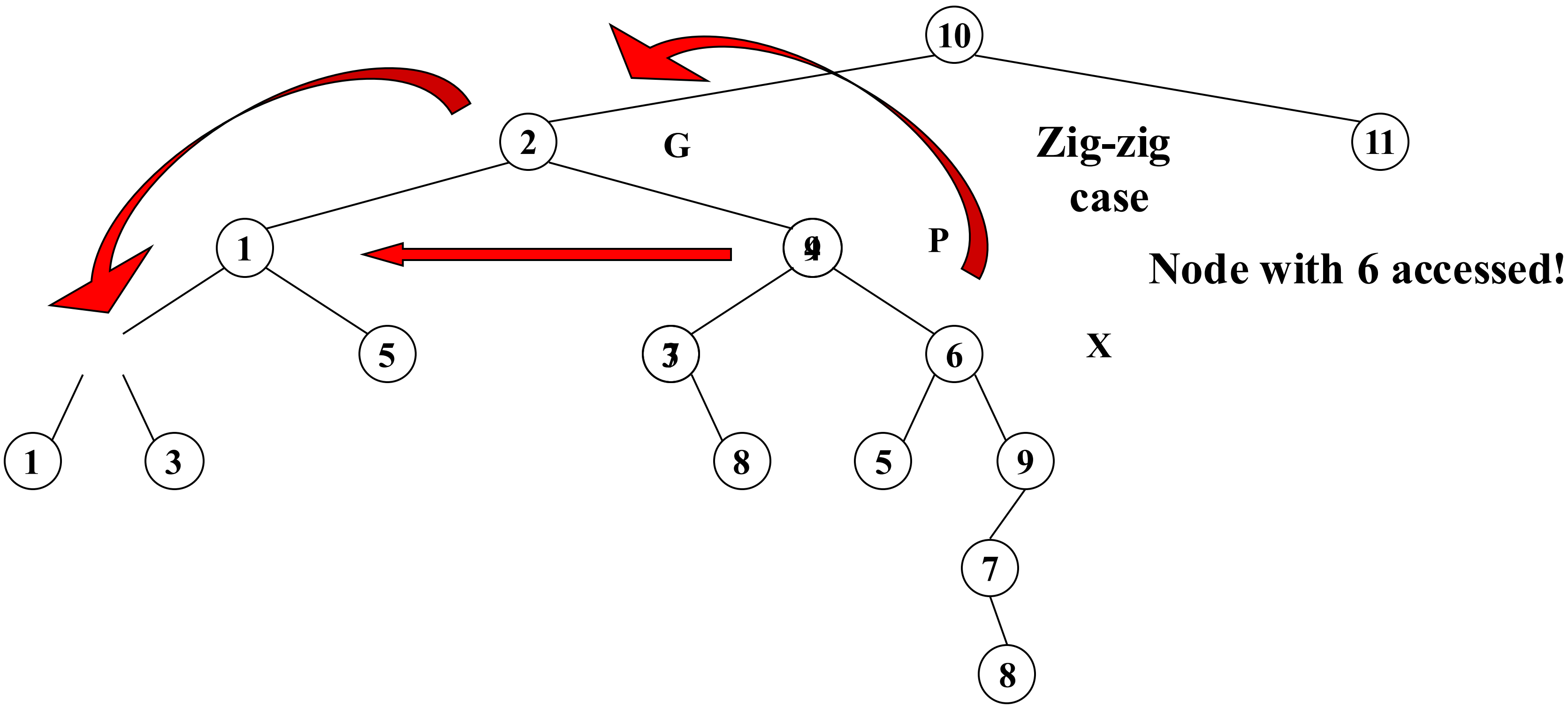
Animated Example



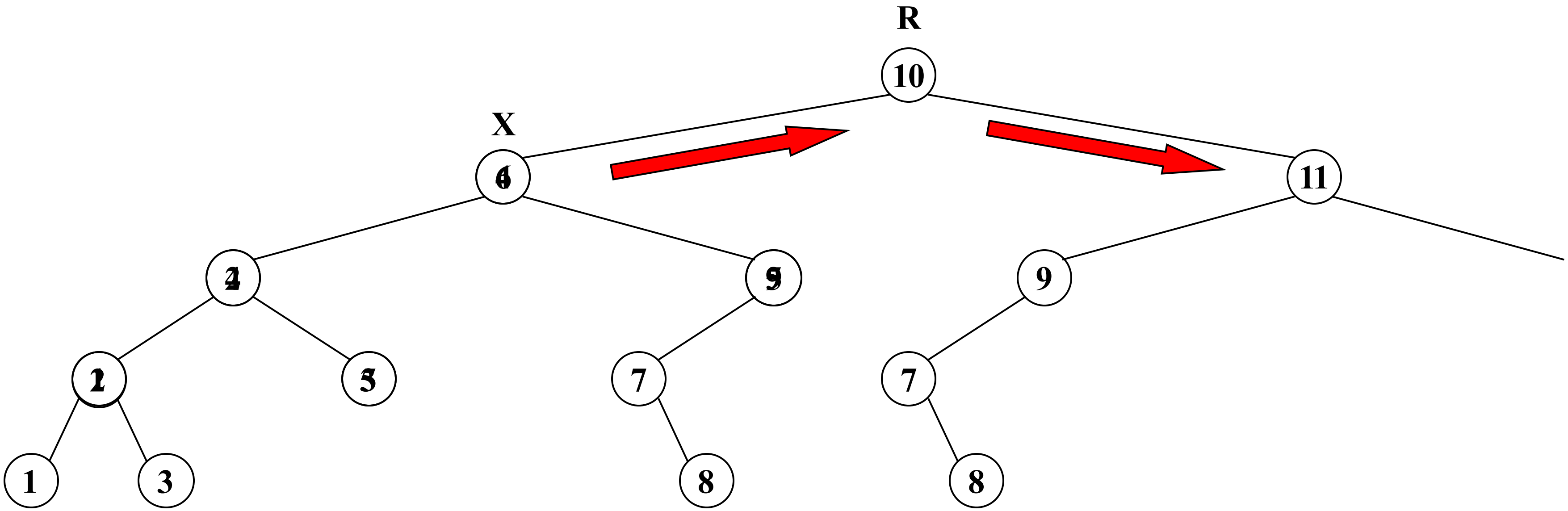
Node with 6 accessed!

Zig-zag case

Animated Example

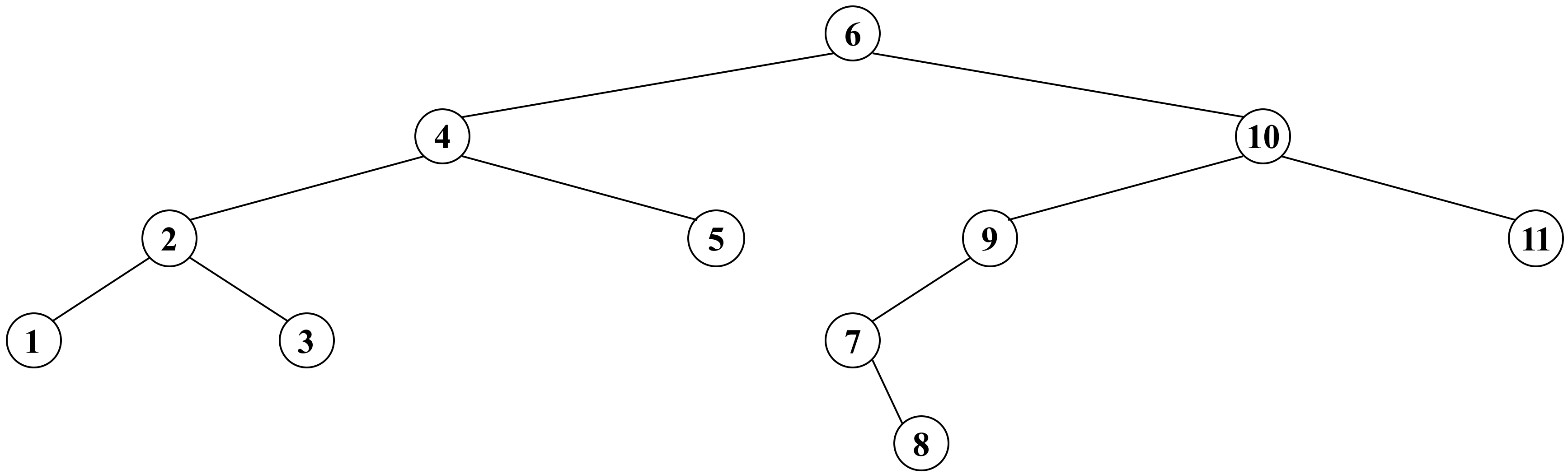


Animated Example



Node with 6 accessed!

Animated Example



Node with 6 accessed!

B-Trees

Motivation for B-Trees

- Two technologies for providing memory capacity in a computer system
  - *Primary* (main) *memory (silicon chips)*
  - *Secondary storage (magnetic disks)*
- Primary memory
  - 5 orders of magnitude (i.e., about 10^5 times) *faster*,
  - 2 orders of magnitude (about 100 times) *more expensive*, and
  - by at least 2 orders of magnitude *less in size*

  than secondary storage due to mechanical operations involved in magnetic disks.

Motivation for B-Trees

- During one disk read or disk write ((4-8.5msec for 7200 RPM sequential disks (not SSDs!)), MM can be accessed about 10^5 times (100 nanosec per access).
- To reimburse (compensate) for this time, at each disks access, *not a single item*, but one or more ***equal-sized** pages* of items (each page 2^{11}-2^{14} bytes) are accessed.
- We need some data structure to store these ***equal sized** pages* in MM.
- *B-Trees*, with their ***equal-sized leaves** (as big as a page),* are suitable data structures for storing and performing regular operations on paged data.

B-Trees

- A *B-tree* is a rooted tree with the following properties:
- Every node x has the following fields:
  - $n[x]$, the number of keys currently stored in x.
  - the $n[x]$ keys themselves, in *non-decreasing order*, so that

$$key_1[x] \leq key_2[x] \leq \ldots \leq key_{n[x]}[x] \;,$$

  - $leaf[x]$, a boolean value, true if x is a leaf.

B-Trees

- *Each internal (non-leaf) node* has *n[x]+1* pointers, $c_1[x],\ldots, c_{n[x]+1}[x]$, to its children. *Leaf nodes* have *no children*, hence no pointers!
- The keys separate the ranges of keys stored in each subtree: if k_i is any key stored in the subtree with root $c_i[x]$, then

$$k_1 \le key_1[x] \le k_2 \le key_2[x] \le \ldots \le key_{n[x]}[x] \le k_{n[x]+1} .$$

- *All leaves have the same depth*, *h*, equal to the *tree's height*.

B-Trees

- There are lower and upper bounds on the number of keys a node may contain. These bounds can be expressed in terms of a fixed integer $t \geq 2$ called the *minimum degree* of the B-Tree.
  - Lower limits
    - All *nodes but the root* has *at least t-1* keys.
    - Every *internal node but the root* has *at least **t** children*.
    - A non-empty tree's **root** must have *at least one key*.

B-Trees

- Upper limits
  - Every *node* can contain *at most* ***2t-1 keys***.
  - Every *internal node* can have *at most* ***2t children***.
  - A node is defined to be full if it has exactly *2t-1 keys*.
- For a *B-tree* of minimum degree $t \geq 2$ and n nodes

$$h \leq \log_t \frac{n+1}{2}$$

Basic Operations on B-Trees

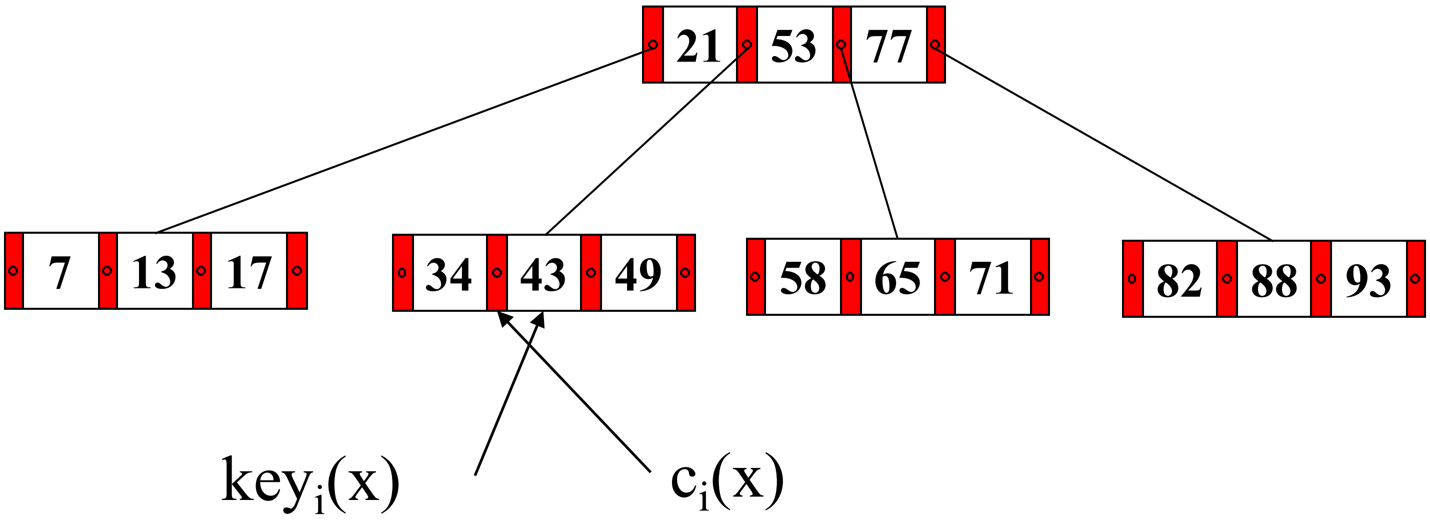
- *B-tree search*
- *B-tree insert*
- *B-tree removal*

Disk Operations in B-Tree operations

- Suppose *x* is a pointer to an object.
- It is accessible if it is in the main memory.
- If it is on the disk, it needs to be transferred to the main memory to be accessible. This is done by *DISK_READ(x)*.
- To save any changes made to any field(s) of the object pointed to by *x*, a *DISK_WRITE(x)* operation is performed.

Search in B-Trees

- Similar to search in BSTs with the exception that instead of a binary, a multi-way (*n[x]*+1-way) decision is made.



Search in B-Trees

```
B-tree-Search(x,k)
{  i=1;
   while (i ≤ n[x] and k > key_i[x]) i++;
   if (i ≤ n[x] and k = key_i[x])             // if key found
        return (x,i);
   if (leaf[x])                        // if key not found at a leaf
        return NULL;
   else {DISK_READ(c_i[x]);                  // if key < key_i[x]
         return B-tree-Search(c_i[x],k);}
}
```

Insertion in B-Trees

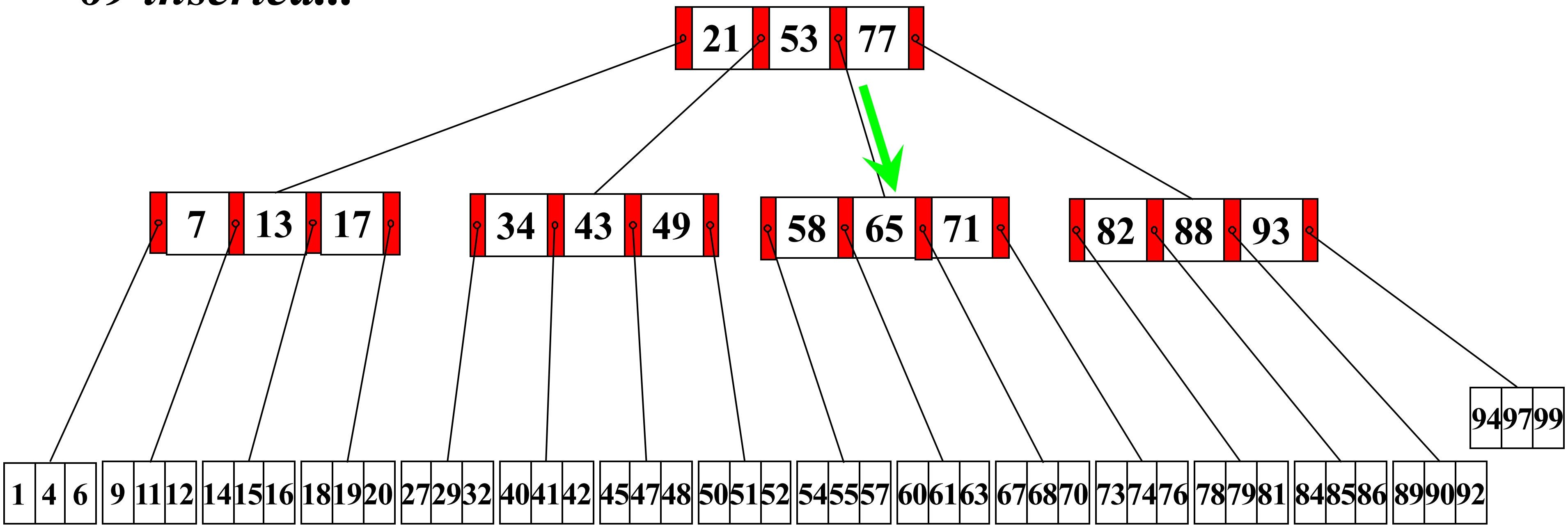
- Insertion into a B-tree is more complicated than that into a BST, since the creation of a new node to place the new key may violate the B-tree property of the tree.
- Instead, the key is put *into a leaf node x if it is not full*.
- If full, a *split* is performed, which splits a full node (with *2t-1* keys) at its *median key*, $key_t[x]$, into two nodes with *t-1* keys each.
- $key_t[x]$ moves up into the parent of *x* and identifies the split point of the two new trees.

Insertion in B-Trees

- A *single-pass insertion* starts at the root traversing *down to the leaf* into which the key is to be inserted.
- On the path down, *all full nodes are split* including a full leaf that also guarantees a parent with an available position for the median key of a full node to be placed.

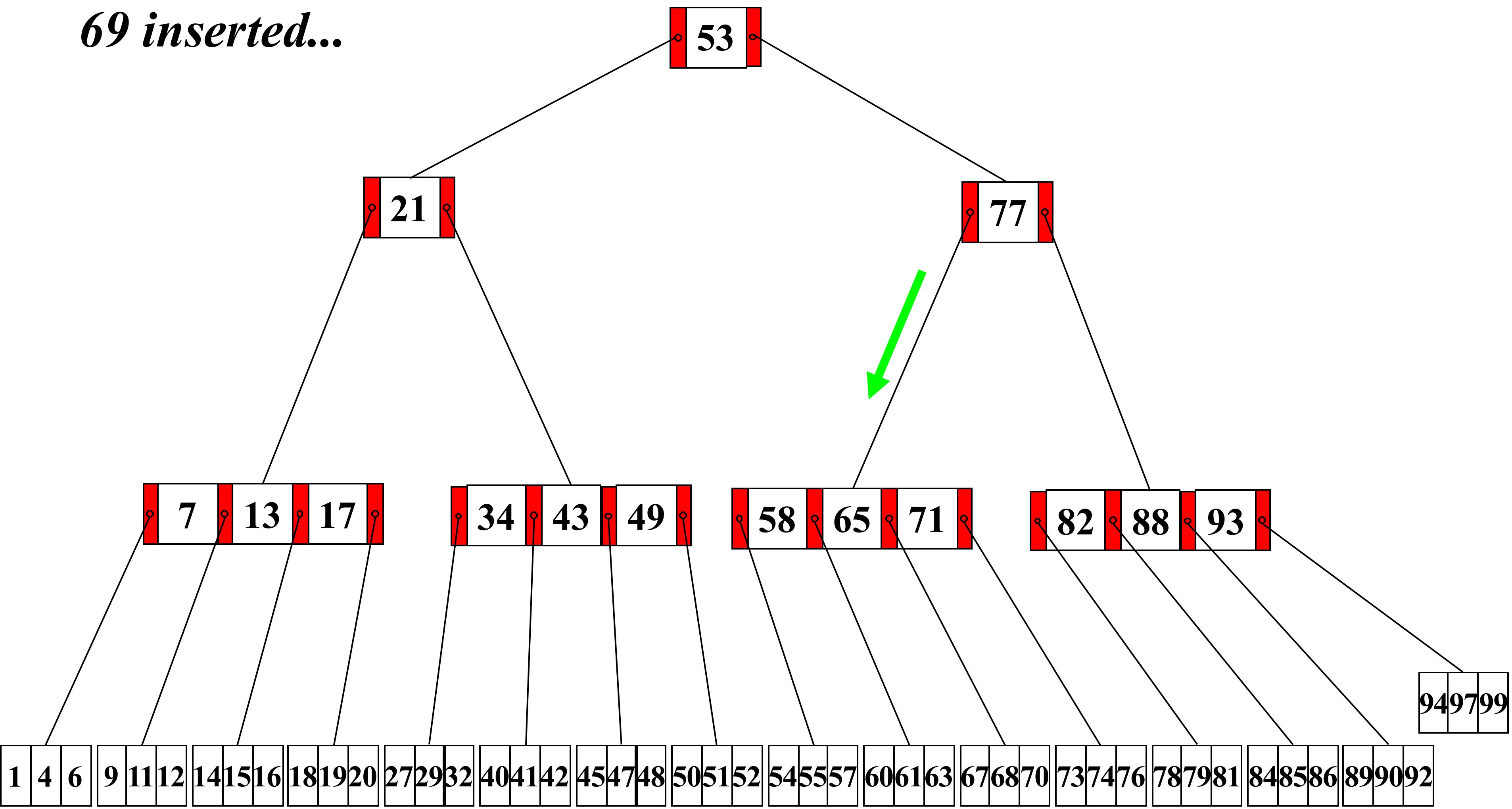
Insertion in B-Trees: Example

69 inserted...



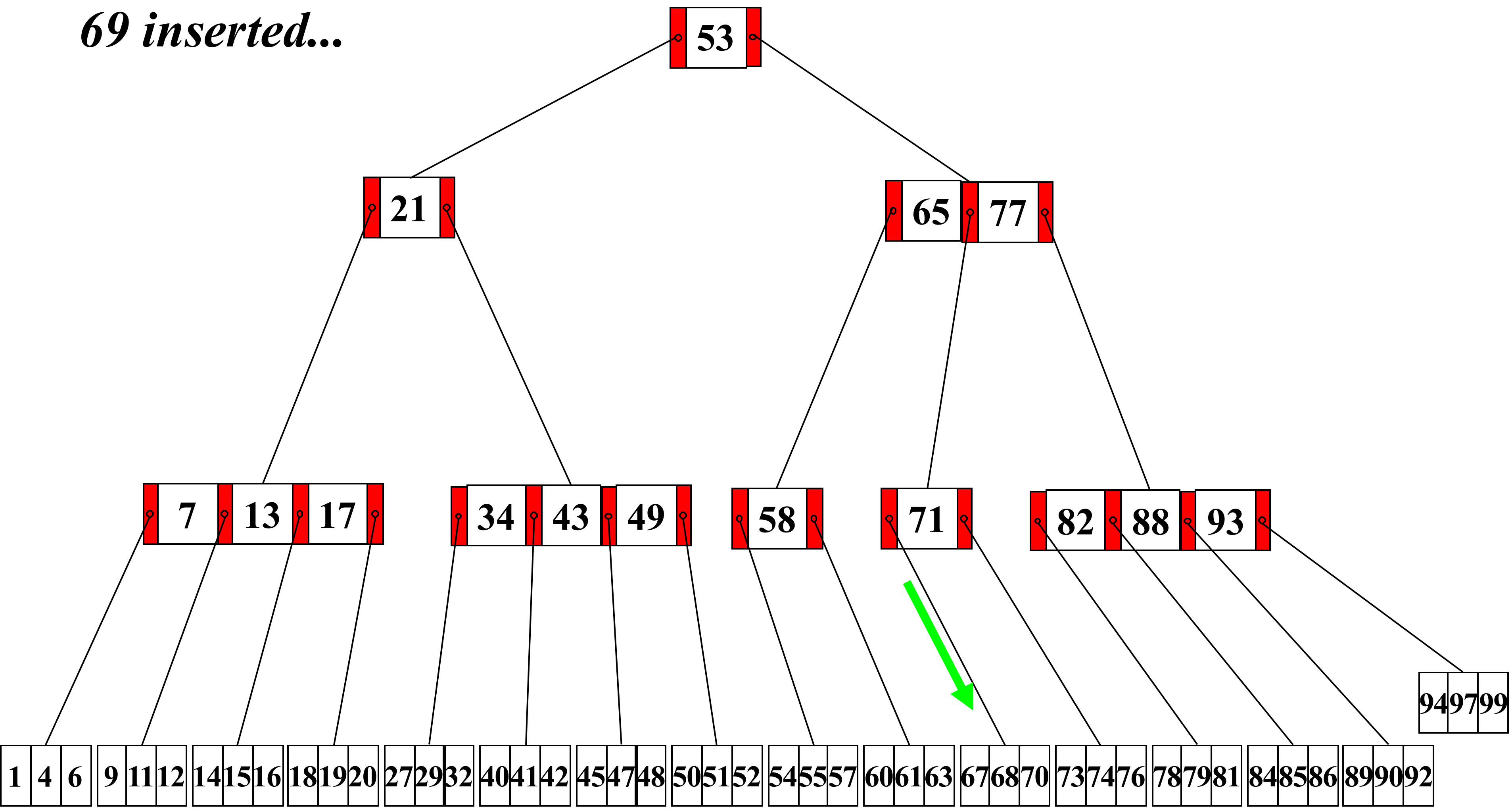
Insertion in B-Trees: Example

69 inserted...



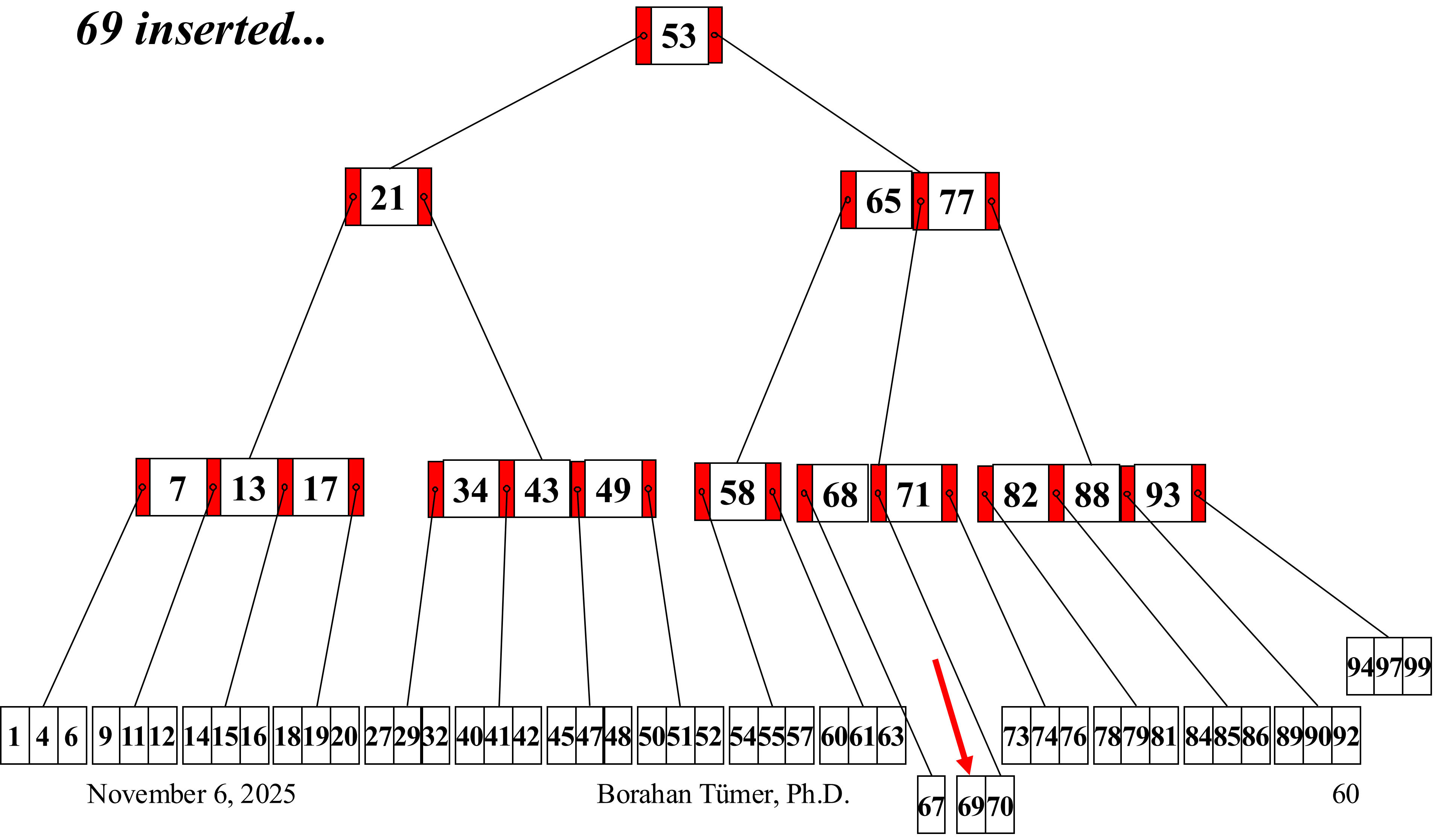
Insertion in B-Trees: Example

69 inserted...



Insertion in B-Trees: Example

69 inserted...



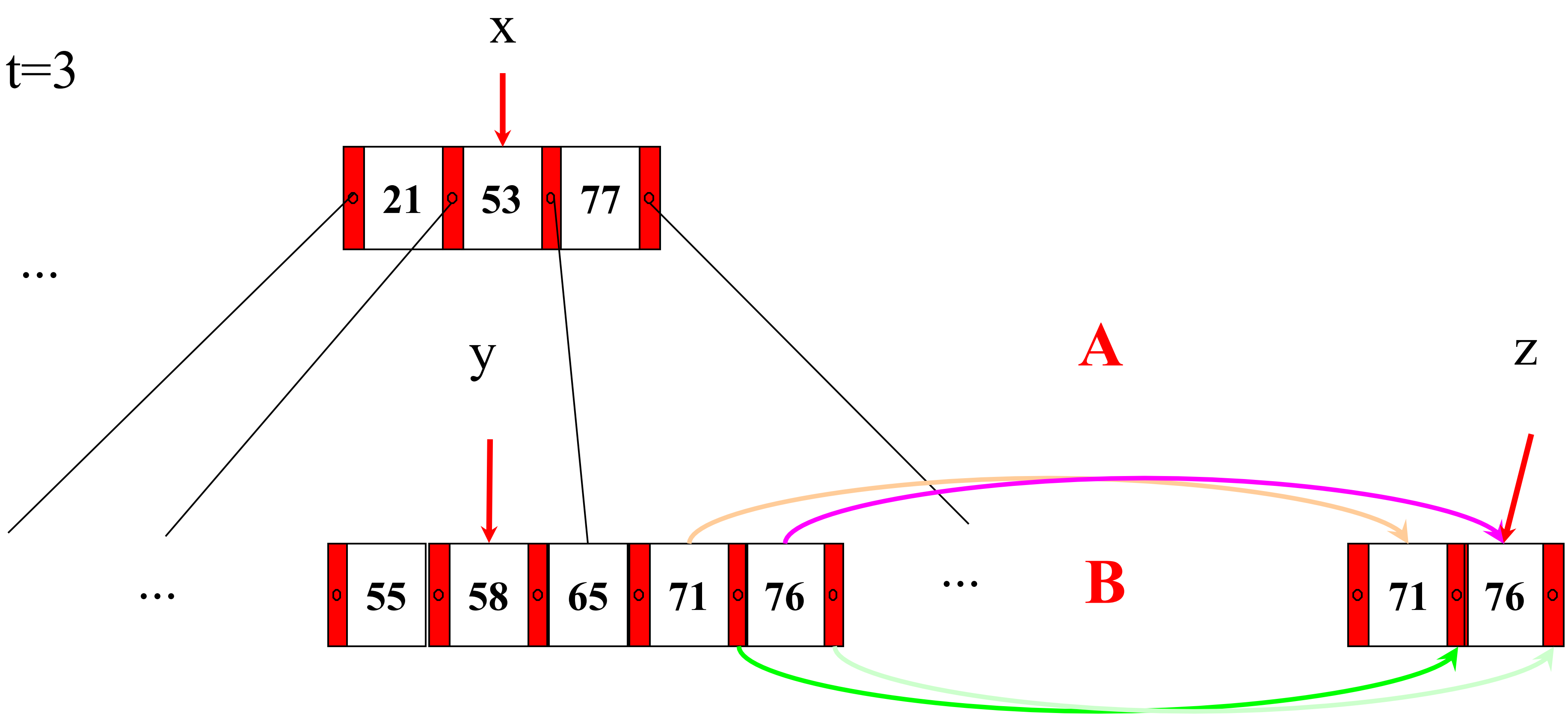
Insertion in B-Trees:B-tree-Insert

```
B-tree-Insert(T,k)
{  r=root[T];
   if (n[r] == 2t-1) {
        s=malloc(new-B-tree-node);
        root[T]=s;
        leaf[s]=false;
        n[s]=0;
        c1[s]=r;
        B-tree-Split-Child(s,1,r);
        B-tree-Insert-Nonfull(s,k); }
   else B-tree-Insert-Nonfull(r,k);
}
```

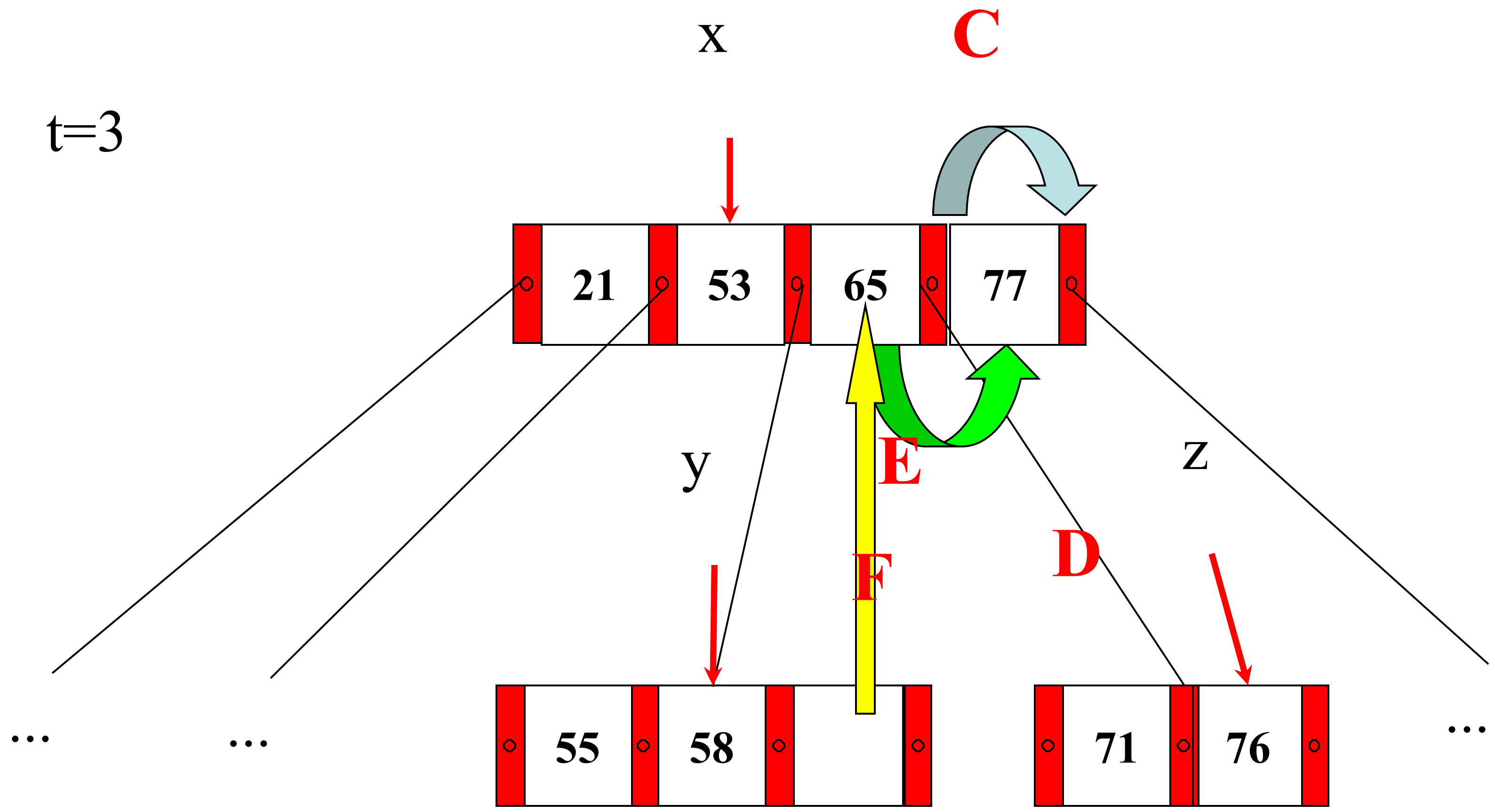
Insertion in B-Trees: B-tree-Split-Child

```
B-tree-Split-Child(x,i,y)
{   z=malloc(new-B-tree-node);
    leaf[z]=leaf[y];
    n[z]=t-1;
    for (j = 1; j < t; j++) key_j[z]=key_{j+t}[y];          A //y's 2nd half of keys to z
    if (!leaf[y])
        for (j = 1; j <= t;j++) c_j[z]=c_{j+t}[y];          B //y's 2nd half of pointers to z
    n[y]=t-1;
    for (j=n[x]+1; j>=i+1; j--) c_{j+1}[x]=c_j[x];          C //shift pointers right
    c_{i+1}[x]=z;                                           D //z pointed by i+1st pntr of x
    for (j=n[x]; j>=i; j--) key_{j+1}[x]=key_j[x];          E //shift keys in x right
    key_i[x]=key_t[y]; n[x]++;                              F //median key moved to parent
    DISK_WRITE(y);
    DISK_WRITE(z);
    DISK_WRITE(x);
}
```

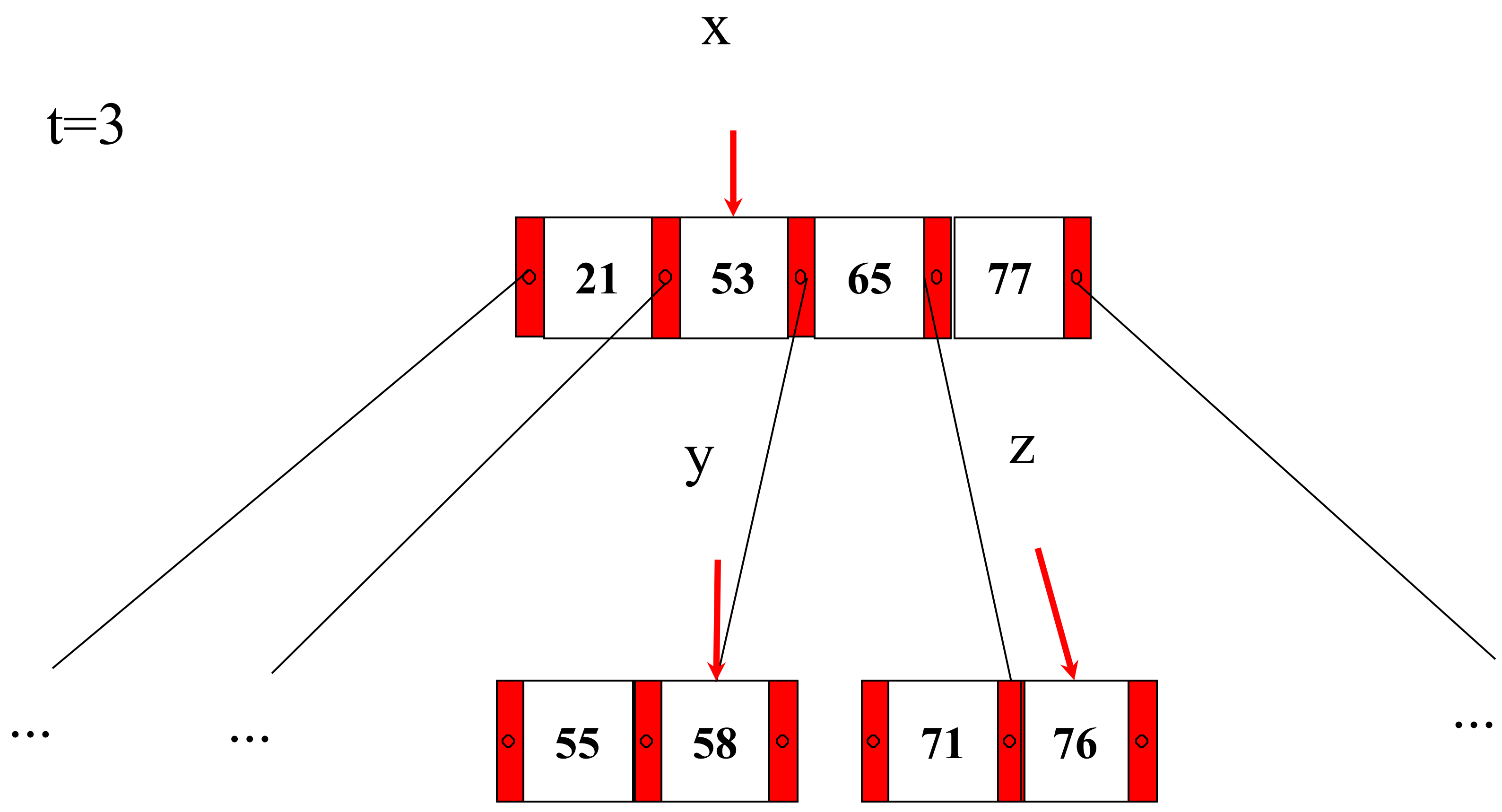
B-tree-Split-Child: Example



B-tree-Split-Child: Example



B-tree-Split-Child: Example



Insertion in B-Trees: B-tree-Insert-Nonfull

```
B-tree-Insert-Nonfull(x,k)
{   i=n[x];
    if (leaf[x]) {
            while (i≥1 and k < key_i[x]) {key_{i+1}[x]=key_i[x]; i--;}
            key_{i+1}[x]=k;
            n[x]++;
            DISK_WRITE(x);
    }
    else {
            while (i≥1 and k < key_i[x]) i--;
            i++;
            DISK_READ(c_i[x]);
            if (n[c_i[x]]==2t-1) {
                    B-tree-Split-Child(x,i, c_i[x]);
                    if (k > key_i[x]) i++;
            }
            B-tree-Insert-Nonfull(c_i[x],k);
        }
}
```

if x is a leaf
then place key in x;
write x on disk;
else find the node (root of subtree) key goes to;
read node from disk;
if node full
split node at key's position;
recursive call with node split and key;

Removing a key from a B-Tree

- Removal in B-trees is different than insertion only in that *a key may be removed from any node, not just from a leaf*.
- As the insertion algorithm splits any full node down the path to the leaf to which the key is to be inserted, a recursive removal algorithm may be written to ensure that for any call to removal on a node x, the number of keys in x is at least the minimum degree t.

Various Cases of Removing a key from a B-Tree

1. If the key k is in node x and x is a leaf, remove the key k from x.
2. If the key k is in node x and x is an internal node, then
   a. If the child y that precedes k in node x has at least t keys, then find the predecessor k' of k in the subtree rooted at y. Recursively delete k', and replace k by k' in x. Finding k' and deleting it can be performed in a single downward pass.

Various Cases of Removal a key from a B-Tree

b. Symmetrically, if the child *z* that follows *k* in node *x* has at least *t* keys, then find the successor *k'* of *k* in the subtree rooted at *z*. Recursively delete *k'*, and replace *k* by *k'* in *x*. Finding *k'* and deleting it can be performed in a single downward pass.

c. Otherwise, if both *y* and *z* have only *t-1* keys, merge *k* and all of *z* into *y* so that x loses both *k* and the pointer to *z* and *y* now contains *2t-1* keys. Free *z* and recursively delete *k* from *y*.

Various Cases of Removal a key from a B-Tree

3. If k is not present in internal node x, determine root $c_i[x]$ of the subtree that must contain k, if k exists in the tree. If $c_i[x]$ has only $t\text{-}1$ keys, execute step *3a* or *3b* as necessary to guarantee that we descend to a node containing at least t keys. Then finish by recursing on the appropriate child of x.

Various Cases of Removal a key from a B-Tree

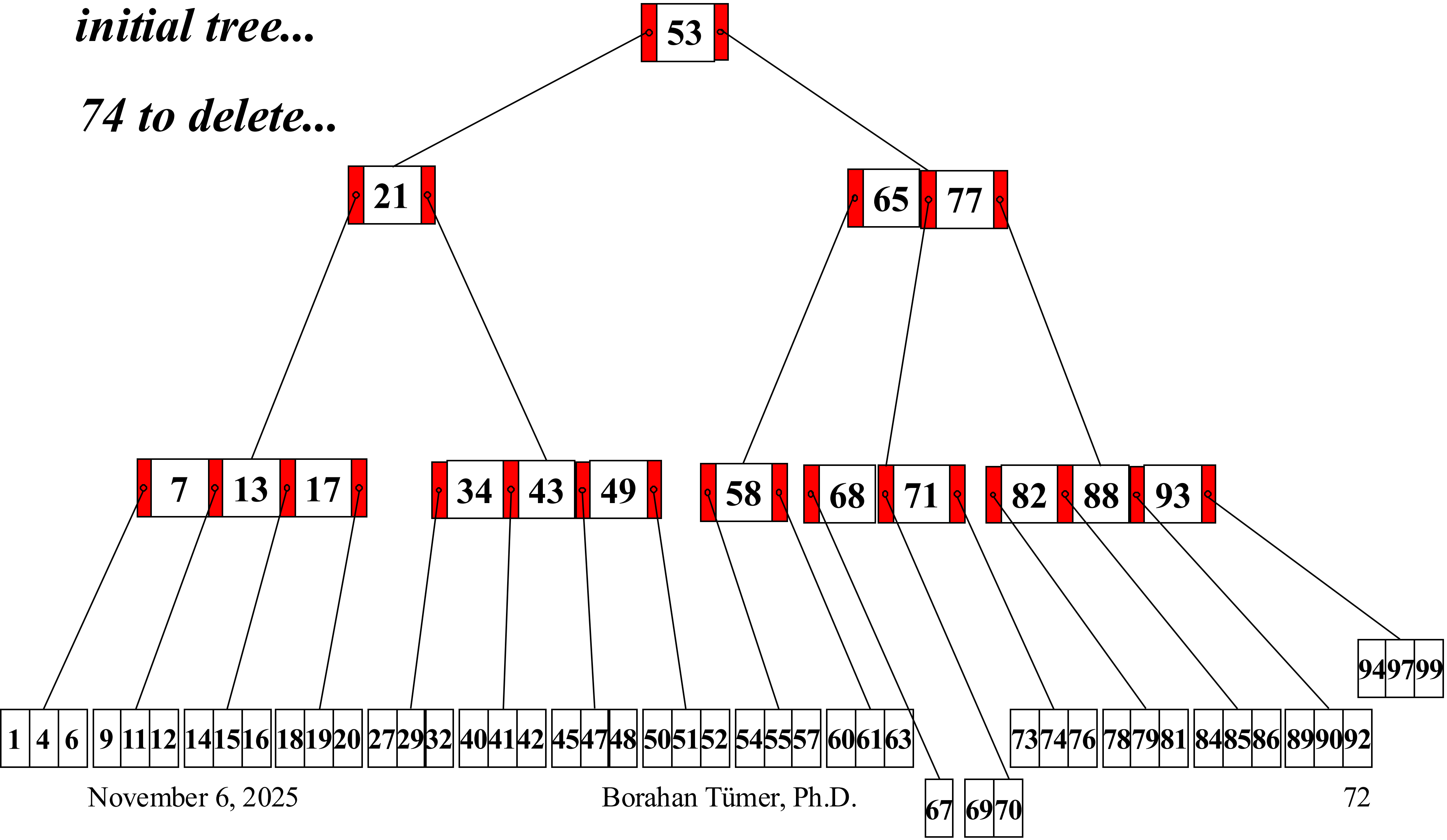
a. If $c_i[x]$ has only t-1 keys but has an immediate sibling with at least t keys, give $c_i[x]$ an extra key by moving a key from x down into $c_i[x]$, moving a key from $c_i[x]$'s immediate left or right sibling up into x, and moving the appropriate child pointer from the sibling into $c_i[x]$.

b. If $c_i[x]$ and both of $c_i[x]$'s immediate siblings have t-1 keys, merge $c_i[x]$ with one sibling, which involves moving a key from x down into the new merged node to become the *median key* for that node.

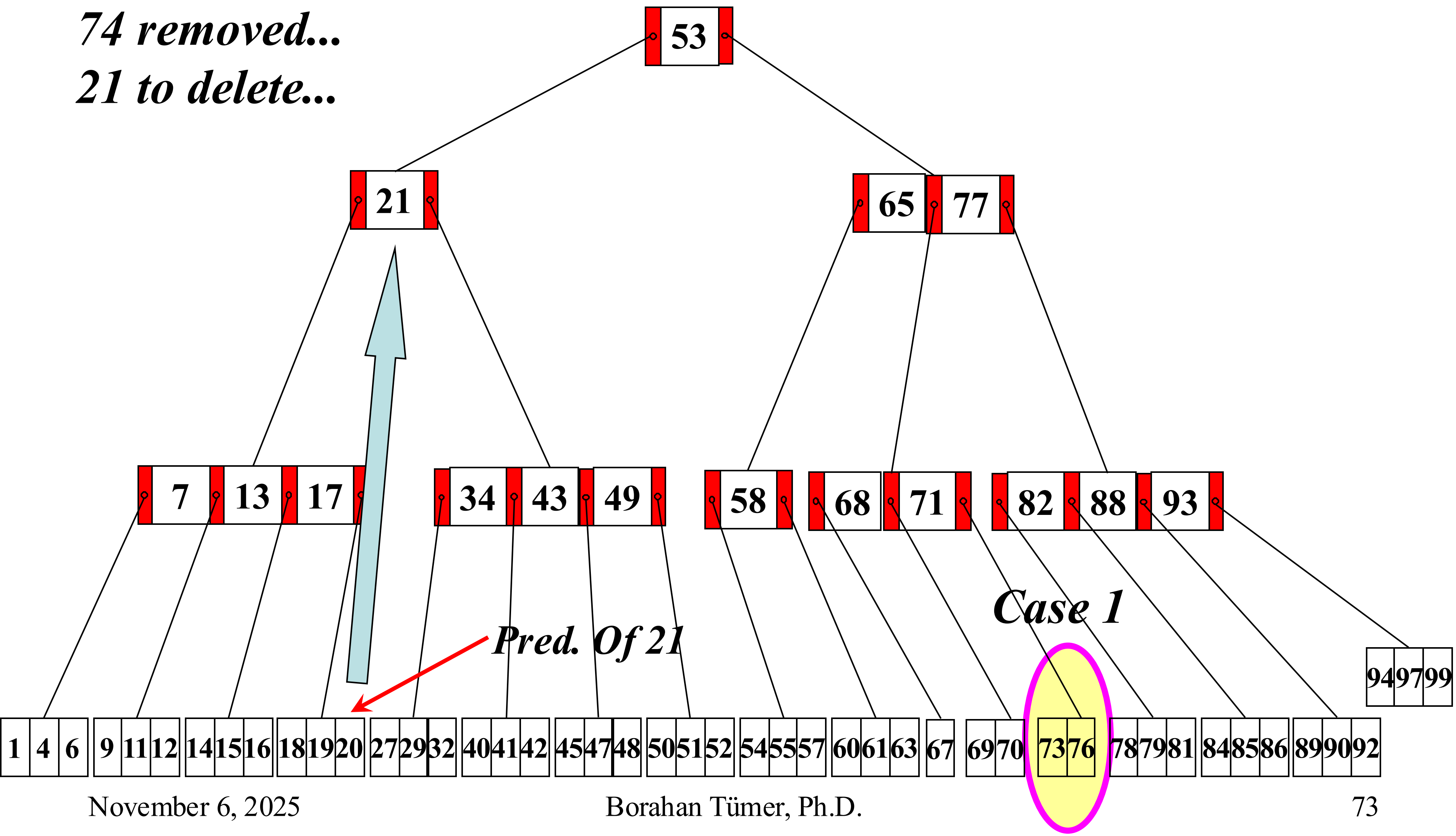
Removal in B-Trees: Example

initial tree...

74 to delete...

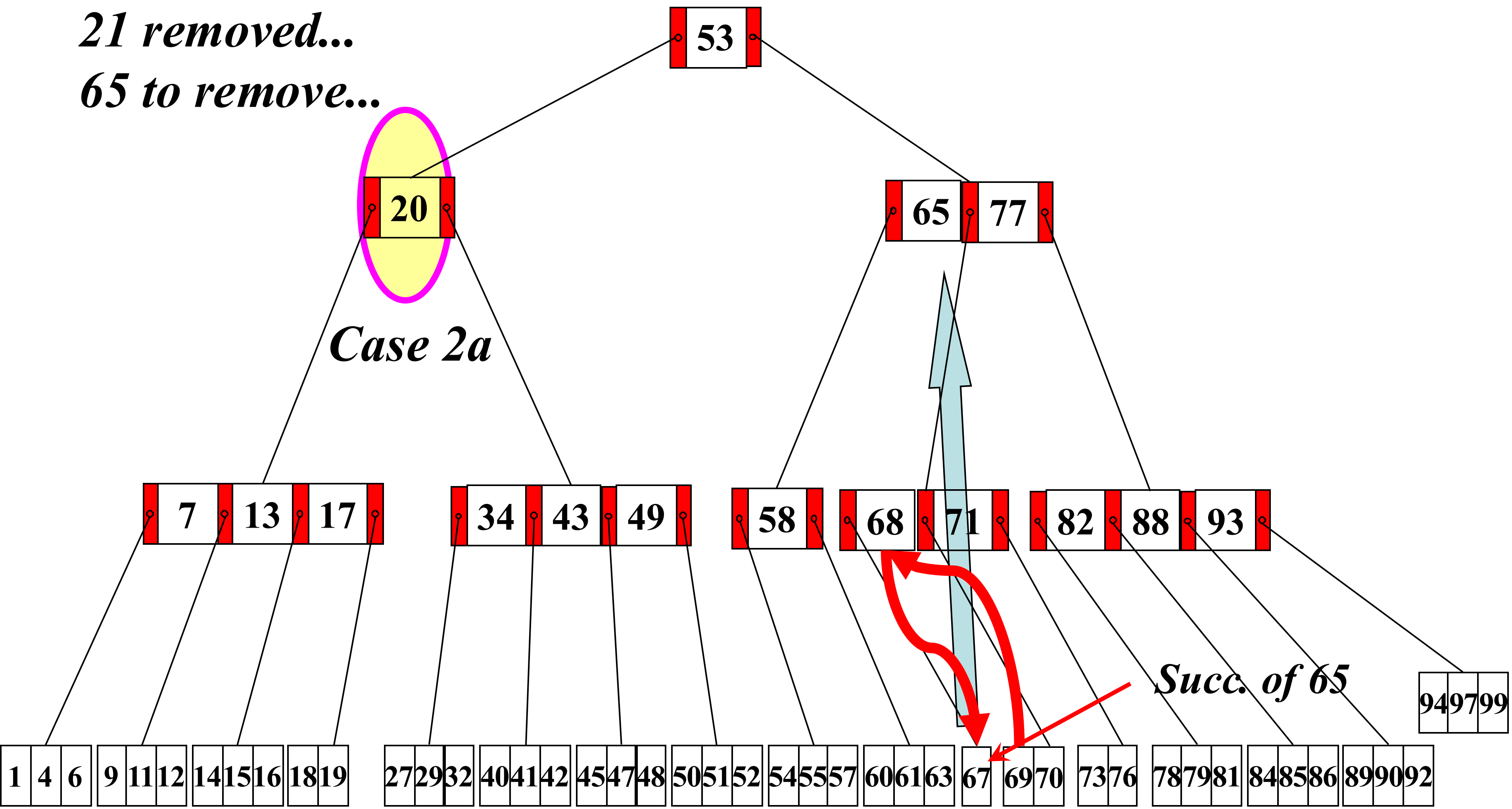


Removal in B-Trees: Example



Removal in B-Trees: Example

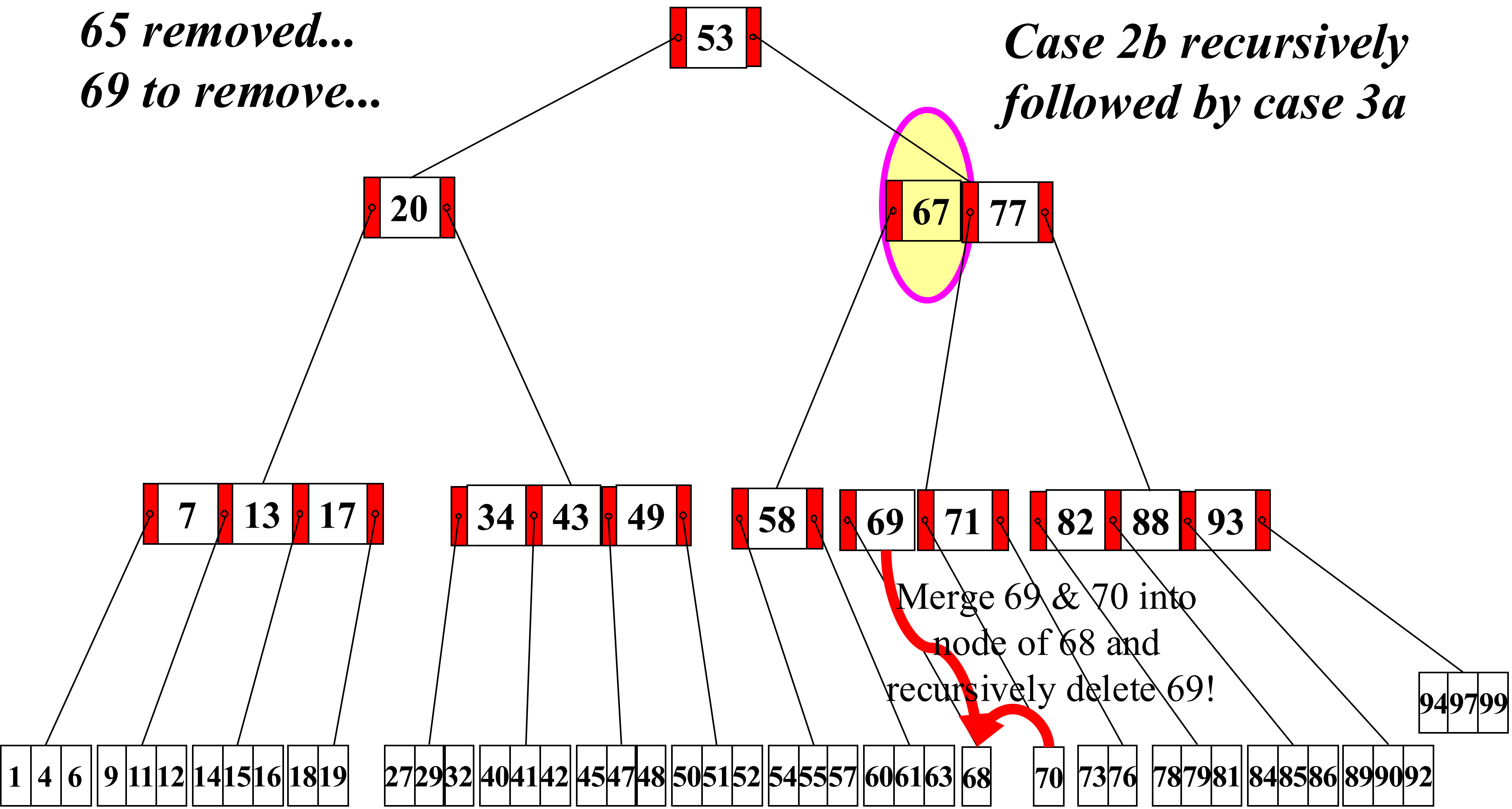
21 removed...
65 to remove...



Removal in B-Trees: Example

65 removed...
69 to remove...

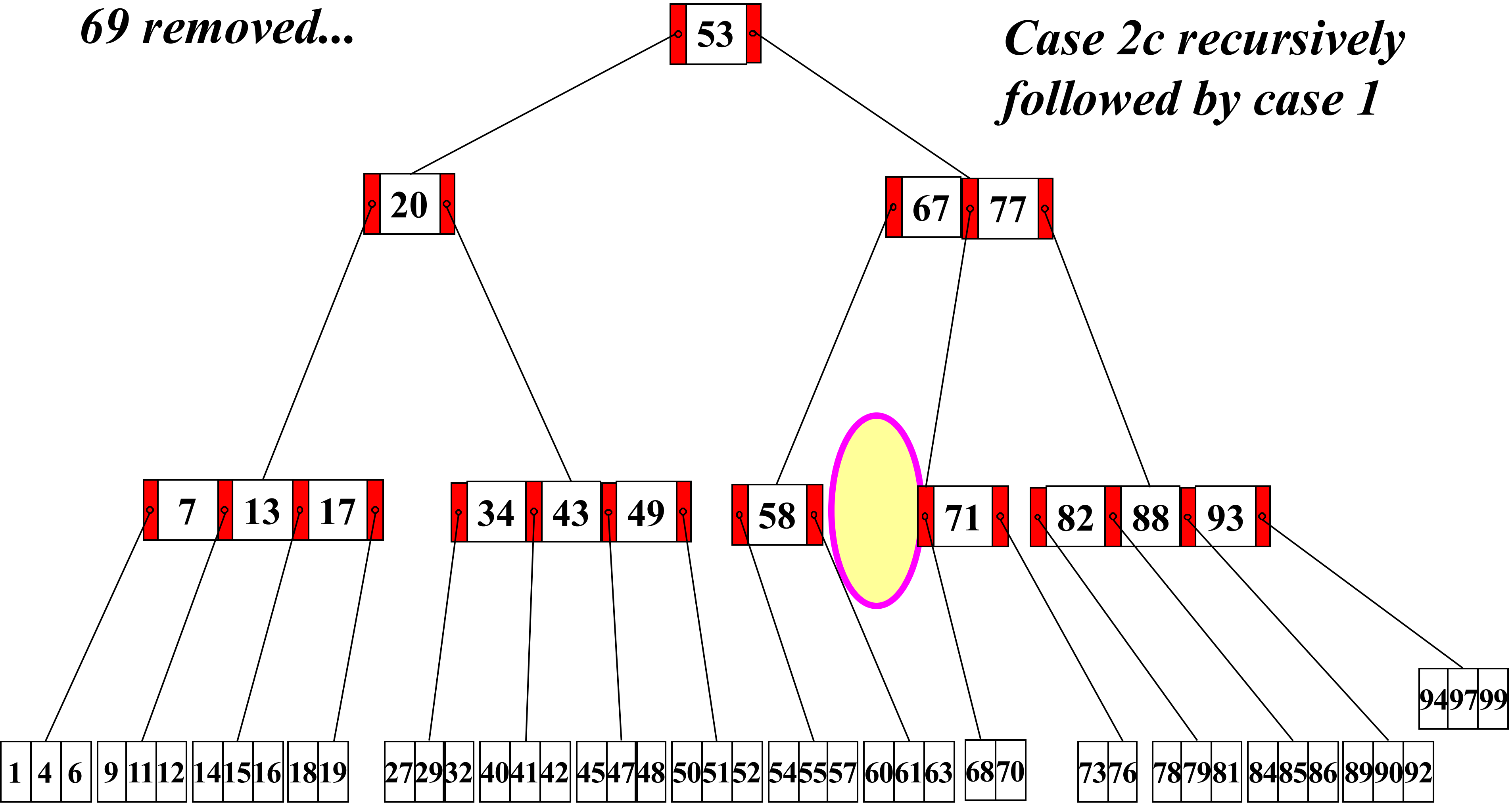
Case 2b recursively followed by case 3a



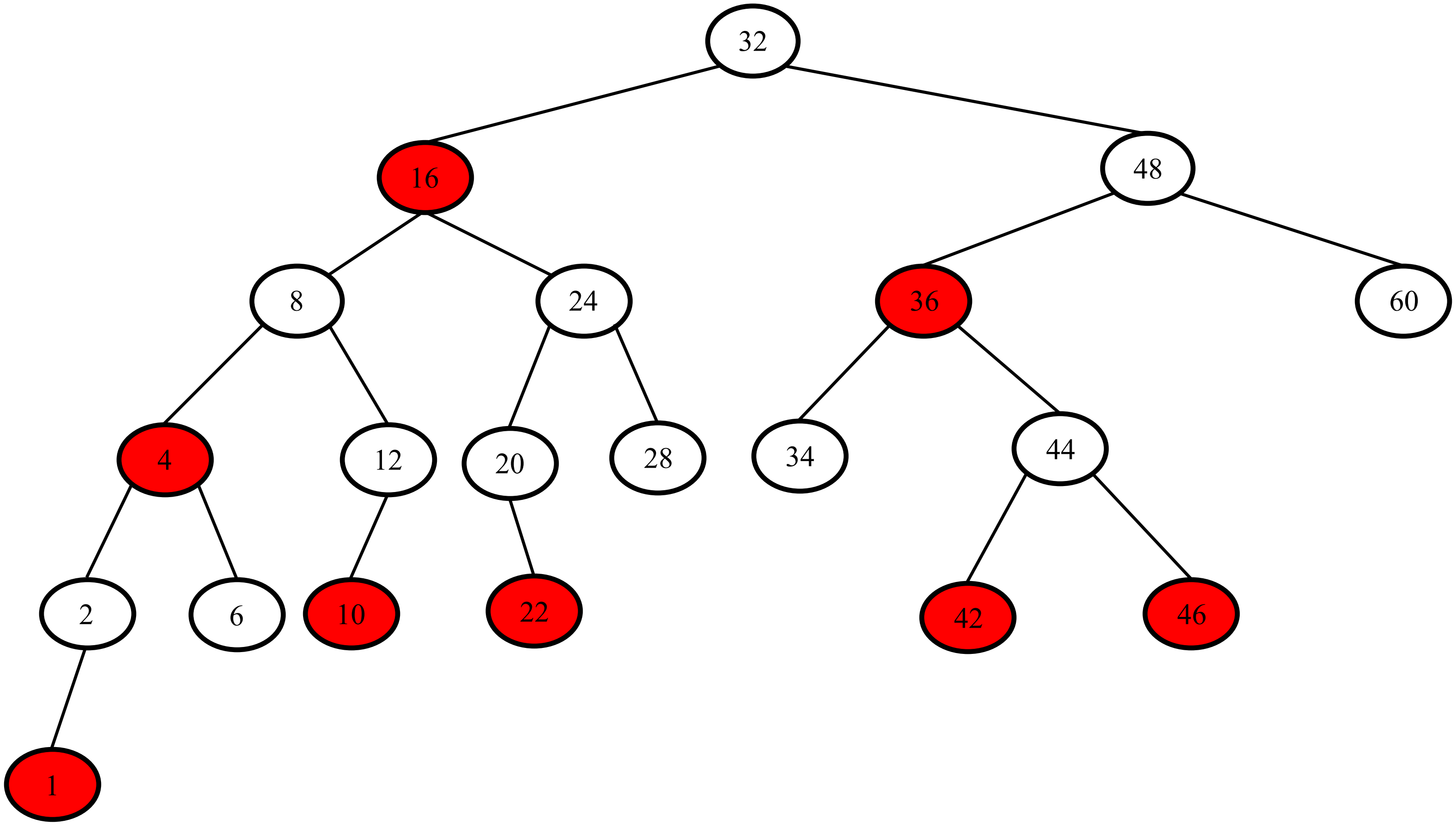
Removal in B-Trees: Example

69 removed...

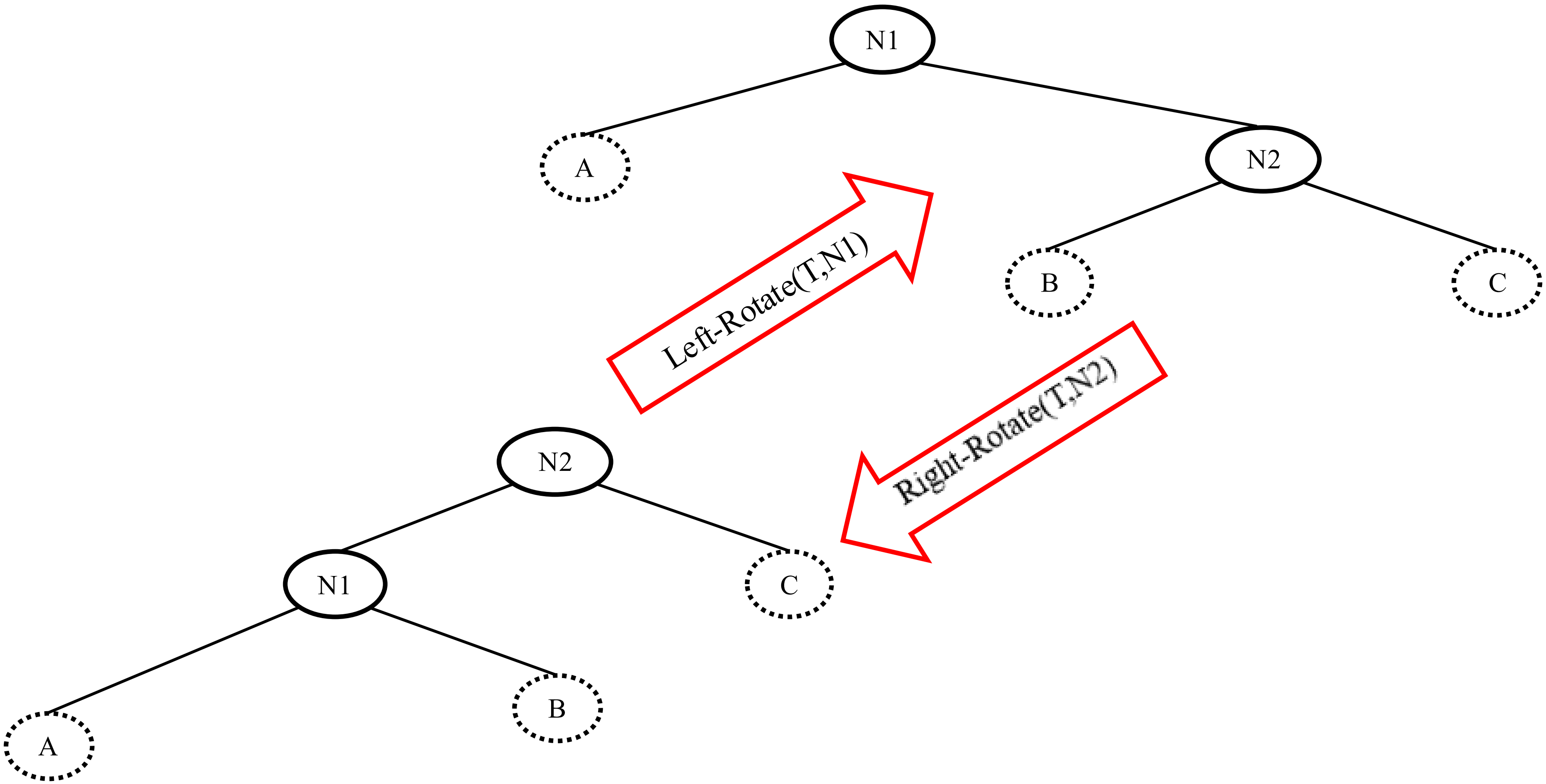
Case 2c recursively followed by case 1



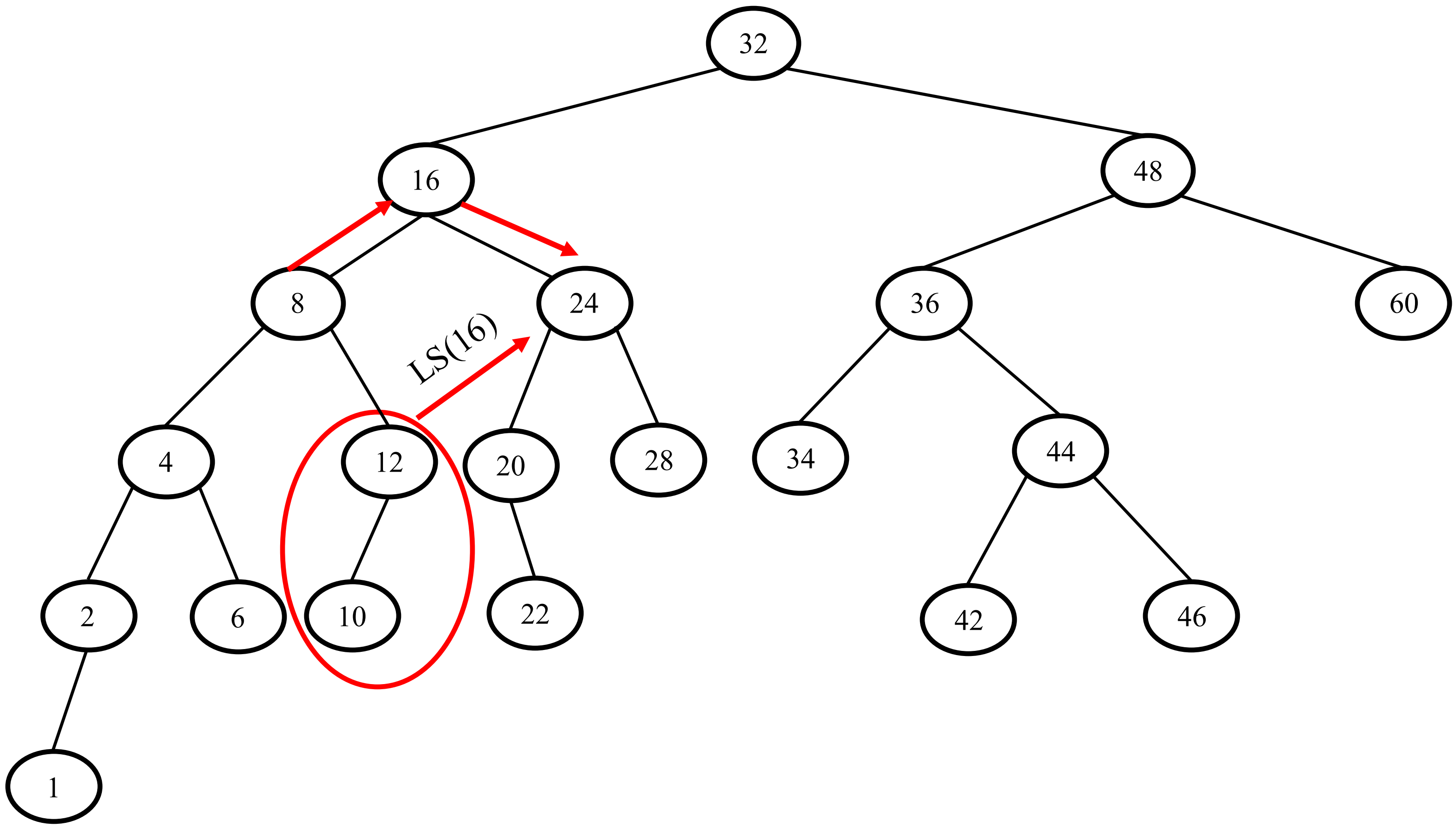
Example RBT



Rotations

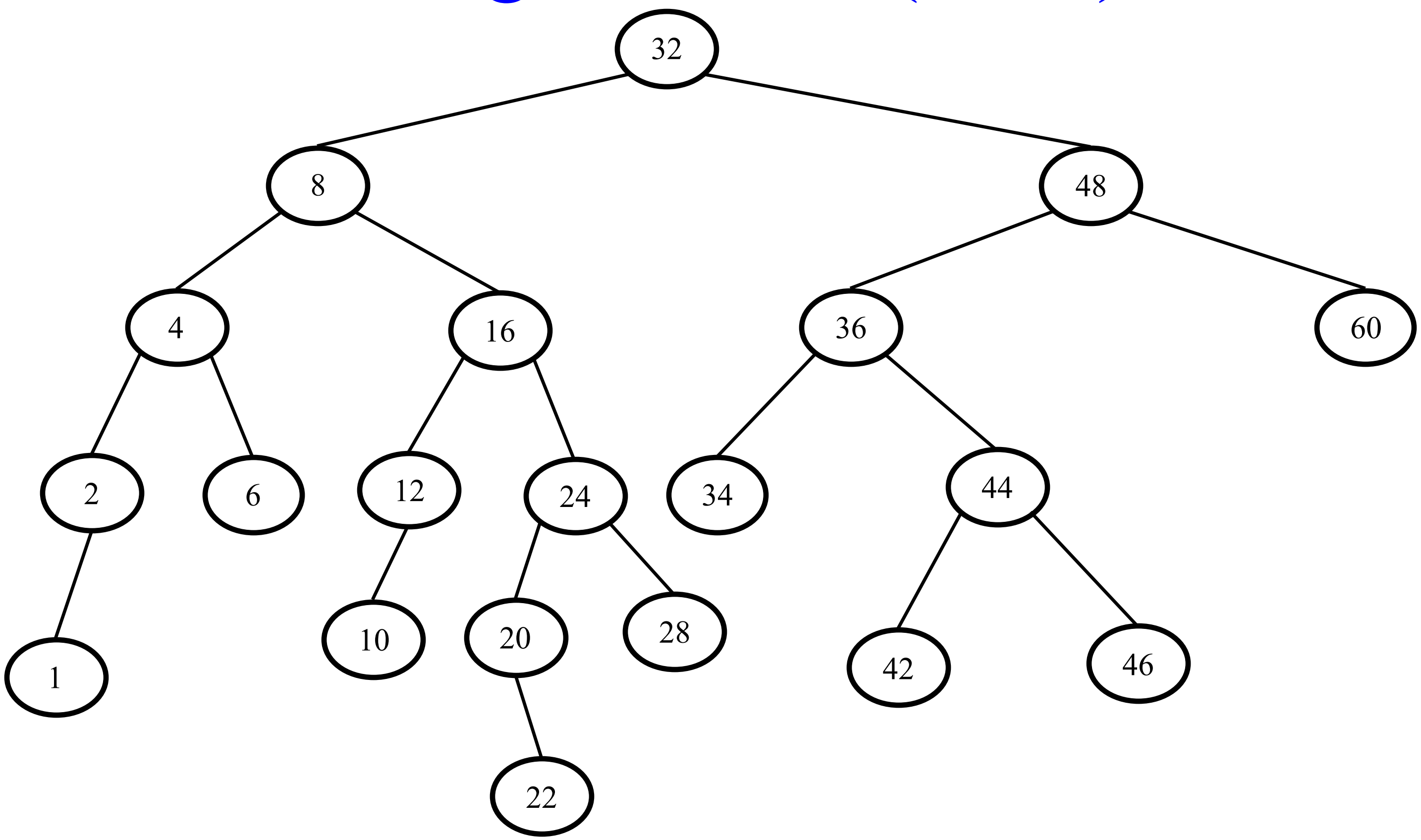


Example RBT
Right-Rotate(T,16)



LS stands for « Left Subtree of »

Example Rotation
Right-Rotate(T,16)



Insertion O(lgn)

- RB-INSERT(T,z)
- /*z inserted to T in $O(\log n)$
- y←nil[T]; x← root[T];
- while x≠ nil[T] do
  - y ←x
  - if (key[z]<key[x])
    - x ←left[x]
  - else x ←right[x]
- p[z]=y
- if y=nil[T]
  - root[T]←z
  - else if (key[z]<key[y])
      - left[y] ←z
    - else right[y] ←z
- left[z] ← nil[T]; right[z] ← nil[T];
- color[z]← RED;
- RB-INSERT-FIXUP(T,z)

Fixing Up Colors after Insertion

- RB-INSERT-FIXUP(T,z)
- while color[p[z]] == RED do
- if (p[z] == left[p[p[z]]])
  - y=right[p[p[z]]];
  - if (color[y]==RED)
    - color[p[z]]=BLACK
    - color[y]=BLACK
    - color[p[p[z]]]=RED
    - z=p[p[z]]

    Case 1
  - else if (z==right[p[z]])
    - z=p[z]
    - LEFT-ROTATE(T,z)

    Case 2
    - color[p[z]]=BLACK
    - color[p[p[z]]]=RED
    - RIGHT-ROTATE(T,p[p[z]])

    Case 3
- else //** *if (p[z] ≠ left[p[p[z]]])*
  - y=left[p[p[z]]];
  - if (color[y]==RED)
    - color[p[z]]=BLACK
    - color[y]=BLACK
    - color[p[p[z]]]=RED
    - z=p[p[z]]

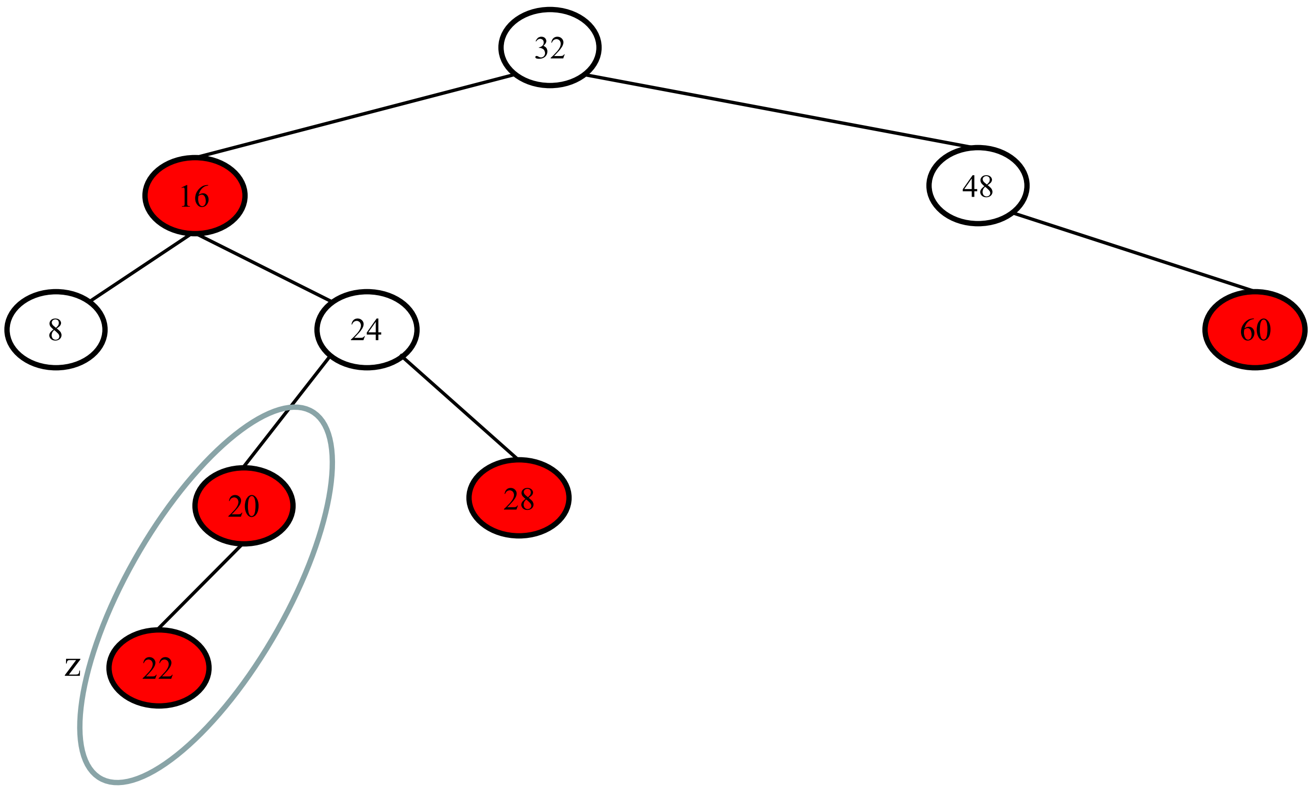
    Case 1
  - else if (z==left[p[z]])
    - z=p[z]
    - RIGHT-ROTATE(T,z)

    Case 2
    - color[p[z]]=BLACK
    - color[p[p[z]]]=RED
    - LEFT-ROTATE(T,p[p[z]])

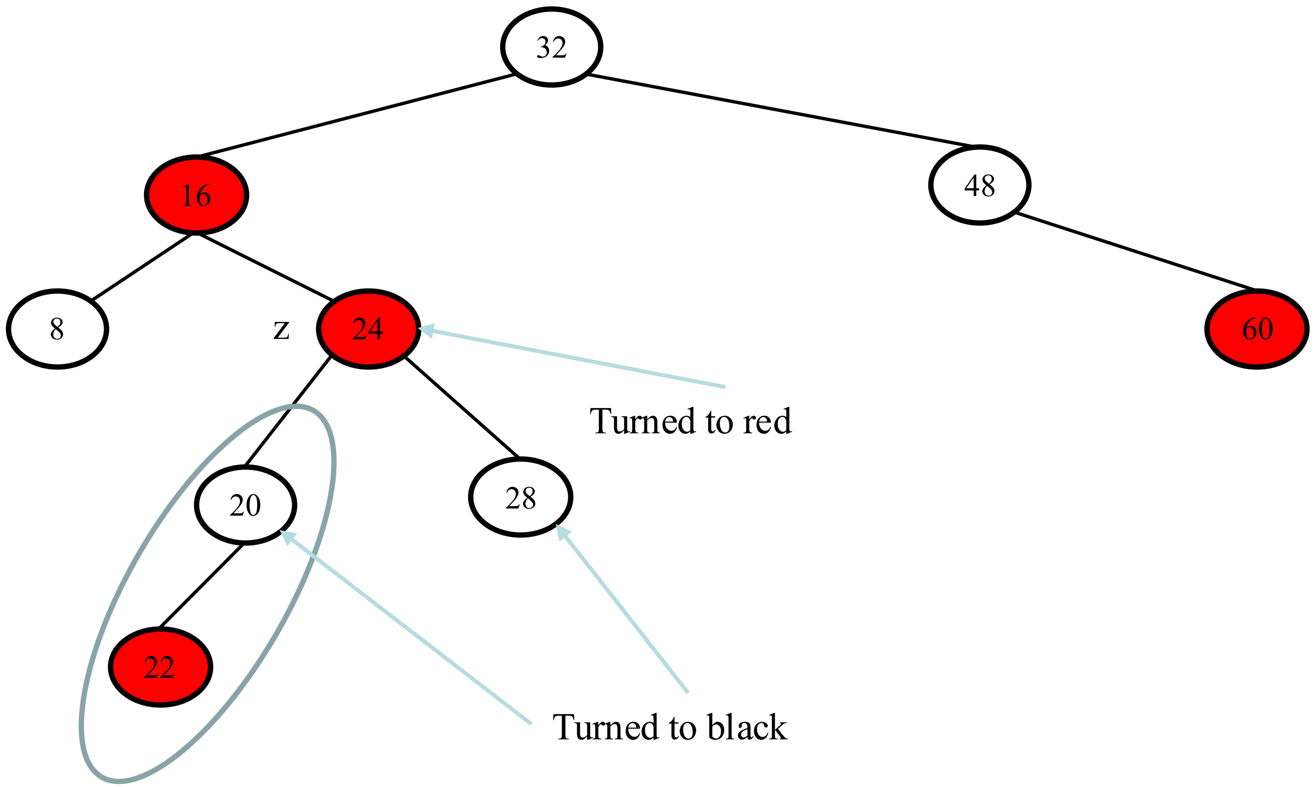
    Case 3
- color[root[T]]=BLACK;

Example: Case 1

Case 1: ***z***'s uncle y is ***red***.

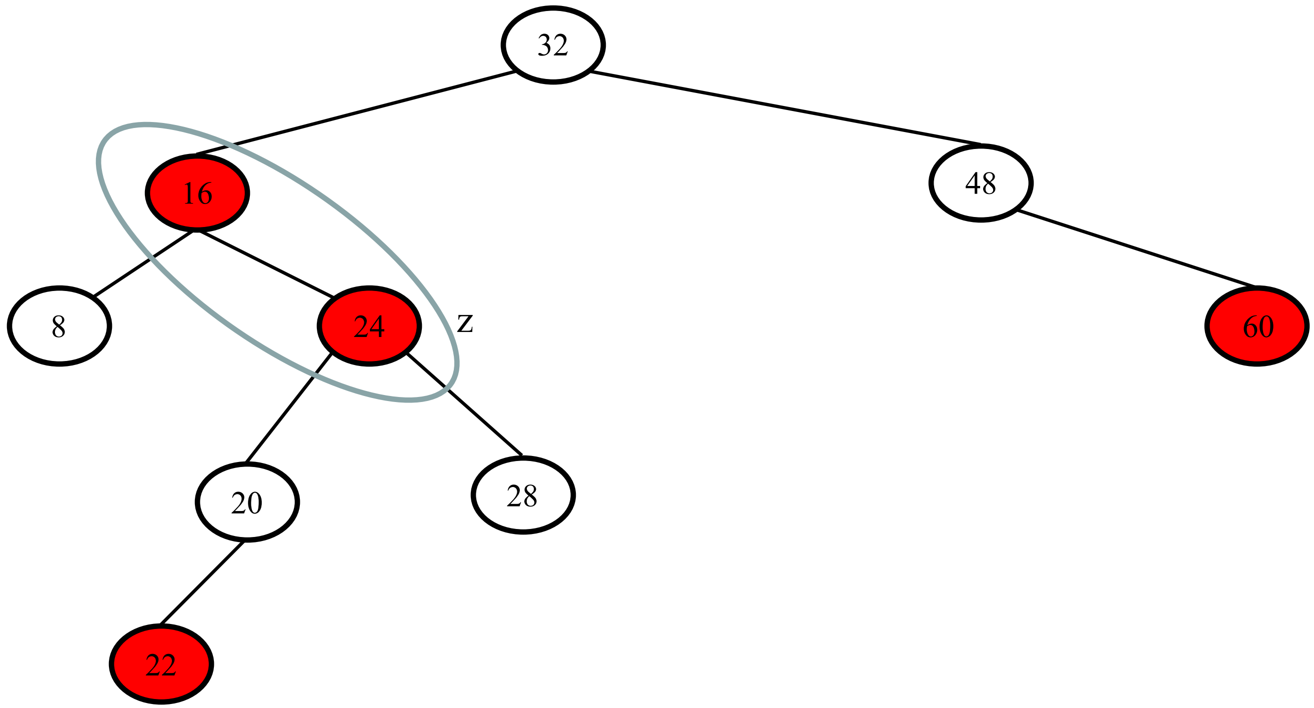


Example: Case 1 solved

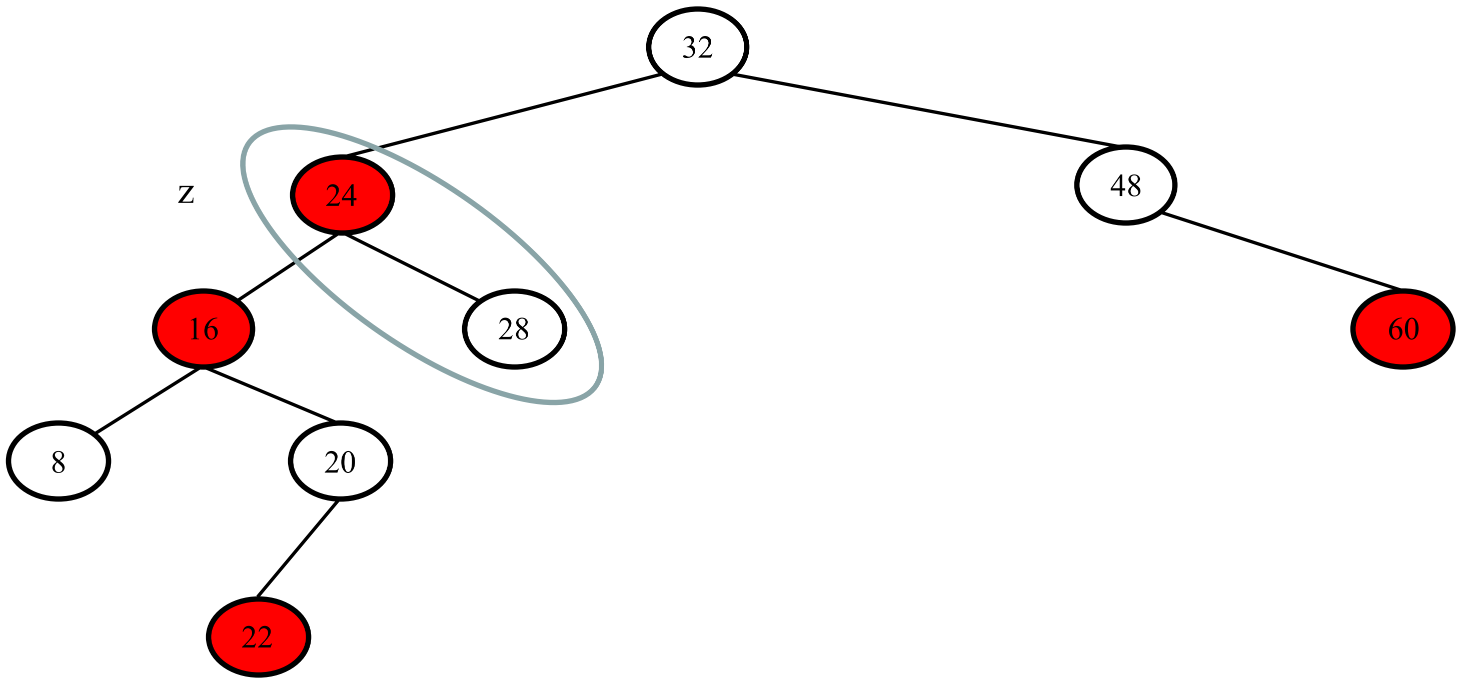


Example: Case 2

Case 2: ***z***'s uncle y is **black** and ***z*** is a right child

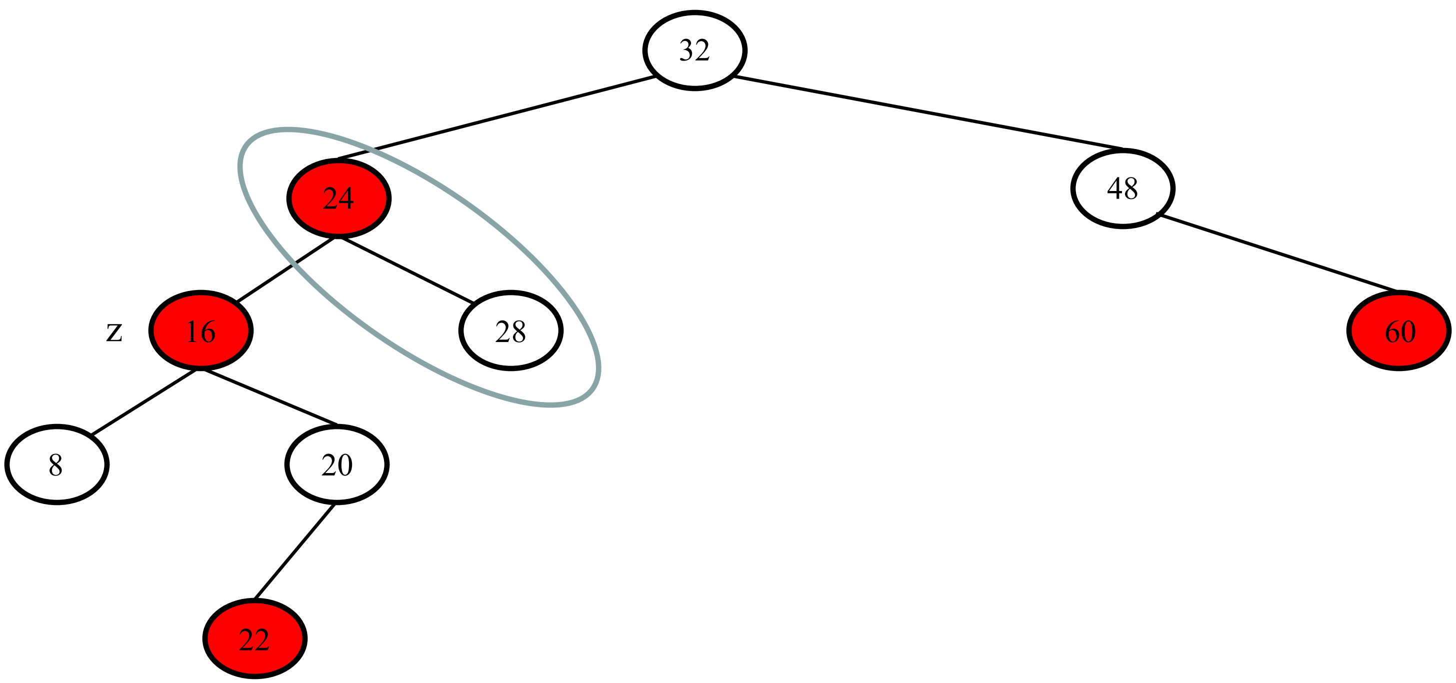


Example: Case 2 solved



Example: Case 3

Case 3: ***z***’s uncle y is **black** and ***z*** is a left child



Example: Case 3 solved